MULTI-DIMENSIONAL COURANT NUMBER INSENSITIVE CE/SE EULER SOLVERS FOR APPLICATIONS INVOLVING HIGHLY NONUNIFORM MESHES

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MULTI-DIMENSIONAL COURANT NUMBER INSENSITIVE CE/SE EULER SOLVERS FOR APPLICATIONS INVOLVING HIGHLY NONUNIFORM MESHES

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Abstract

In a CE/SE simulation with a fixed total marching time, generally the numerical dissipation increases as the value of the CFL number decreases from 1, its maximum stability bound. As such, in a case with a large CFL number disparity (e.g., a simulation with a highly nonuniform spatial mesh and a spatially independent time step), the performance sensitivity with respect to the CFL number can lead to a solution that is highly dissipative in a region where the local CFL number \( \ll 1 \). In this paper, the ideas used in a recent work on one-dimensional CFL number insensitive CE/SE schemes are elaborated in a more detailed manner and also are extended to construct one-dimensional and multidimensional solvers for applications involving nonuniform spatial meshes. Also an error in a recent related paper will be described and corrected. As a by-product of the current work, a new set of wiggle-suppressing weighted-averaging formulae much more potent and flexible than those introduced earlier is also presented.

1. Introduction

The space-time conservation element and solution element (CE/SE) method is a high-resolution and genuinely multidimensional method for solving conservation laws [1–37]. Its nontraditional features include: (i) a unified treatment of space and time; (ii) the introduction of conservation elements (CEs) and solution elements (SEs) as the vehicles for enforcing space-time flux conservation; (iii) a novel time marching strategy that has a space-time staggered stencil at its core and, as such, fluxes at an interface can be evaluated without using any interpolation or extrapolation procedure (which, in turn, leads to the method’s ability to capture shocks without using Riemann solvers); (iv) the requirement that each scheme be built from a non-dissipative core scheme and, as a result, the numerical dissipation can be controlled effectively; and (v) the mesh values of the physical dependent variables and their spatial derivatives are considered as independent marching variables to be solve for simultaneously. Note that CEs are nonoverlapping space-time subdomains introduced such that (i) the computational domain can be filled by these subdomains; and (ii) flux conservation can be enforced over each of them and also over the union of any combination of them. On the other hand, each SE is a space-time subdomain introduced such that (i) the boundary of each CE can be divided into several component parts with each of them belonging to a unique SE; and (ii) within a SE, any physical flux vector is approximated using simple smooth functions. In general, a CE does not coincide with a SE.

Without using preconditioning or other special techniques, since its inception [1] the CE/SE method has been used to obtain numerous highly accurate 1D, 2D and 3D steady and unsteady flow solutions with Mach numbers ranging from 0.0028 to 10 [28]. The flow phenomena modeled include traveling and interacting shocks, acoustic waves, shedding vortices, shock/boundary-layer interaction, detonation waves, cavitation and hydraulic jump [2–37]. In particular, the rather unique capability of the CE/SE method to resolve both strong shocks and small disturbances (e.g., acoustic waves) simultaneously [11,13,14] makes it a unique tool for attacking the problems in computational aeroacoustics (CAA). Note that the fact that the (second order) CE/SE method can solve CAA problems accurately is an exception to the commonly-held wisdom that a second-order scheme is not adequate for solving CAA problems. Also note that, while numerical dissipation is required for shock capturing, it may also result in annihilation of small disturbances. Thus a solver that can handle both strong shocks and small disturbances simultaneously must be able to overcome this difficulty.

In spite of its past successes, there is still room for improving the CE/SE method. An example is the

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fact that, in a CE/SE simulation with a fixed total
marching time, generally the numerical dissipation
increases as the value of the CFL number decreases
from 1, its maximum stability bound. As such, in a
case with a large CFL number disparity (e.g., a simul-
ation with a highly non-uniform spatial mesh
and a spatially independent time step), the per-
formance sensitivity with respect to the CFL number
may lead to a solution that is highly dissipative in a
region where the local CFL number $\ll 1$.

In this paper, the ideas used in a recent work
[35] on one-dimensional CFL number insensitive
CE/SE schemes will be extended to construct one-
dimensional and multidimensional solvers for appli-
cations involving highly nonuniform spatial meshes.

Also a misconception introduced in a recent related
paper [36] will be described and corrected. As a
by-product of the current work, a new set of wiggle-
suppressing weighted-averaging formulae much more
potent and flexible than those introduced earlier [3]
will also be presented. The rest of the paper is
organized as follows. A review of the existing CE/SE
schemes is given in Sec. 2. The new CFL number
insensitive schemes are described in Secs. 3–6.

Numerical results are presented in Sec. 7. Conclu-
sions and discussions are given in Sec. 8.

2. Review of the 1D CE/SE method

For simplicity, we review the existing CE/SE
schemes for the PDE

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$  \quad (2.1)

where $a \neq 0$ is a constant. Let $x_1 = x$, and $x_2 = t$ be
considered as the coordinates of a two-dimensional
Euclidean space $E_2$. Then, by using Gauss’ diver-
gence theorem in the space-time region $E_2$, it can be shown
that Eq. (2.1) is the differential form of the integral
conservation law

$$\oint_{S(V)} \vec{h} \cdot d\vec{s} = 0$$  \quad (2.2)

As depicted in Fig. 1, here (i) $S(V)$ is the bound-
dary of an arbitrary space-time region $V$ in $E_2$, (ii)
$\vec{h} = (au, u)$, and (iii) $d\vec{s} = d\sigma \vec{n}$ with $d\sigma$ and $\vec{n}$,
respectively being the area and the unit outward
normal of a surface element on $S(V)$. Note that:
(i) because $\vec{h} \cdot d\vec{s}$ is the space-time flux of $\vec{h}$
leaving the region $V$ through the surface element $d\vec{s}$,
Eq. (2.2) simply states that the total space-time flux
of $\vec{h}$ leaving $V$ through $S(V)$ vanishes; (ii) in $E_2$, $d\sigma$
is the length of a line segment on the simple closed
curve $S(V)$; and (iii) all mathematical operations
can be carried out as though $E_2$ were an ordinary
two-dimensional Euclidean space.

To proceed, let $\Omega$ denote the set of all space-time
staggered mesh points in $E_2$ (dots in Fig. 2(a)),
where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ and, for each
$n, j = n \pm 1/2, n \pm 3/2, n \pm 5/2, \ldots$. Each $(j, n) \in \Omega$
is associated with a solution element, i.e., $SE(j, n)$.

By definition, $SE(j, n)$ is the interior of the space-
time region bounded by a dashed curve depicted in
Fig. 2(b). It includes a horizontal line segment, a verti-
cal line segment, and their immediate neighborhood.

Let $(x, t) \in SE(j, n)$. Then Eq. (2.2) will be simu-
lated numerically assuming that $u(x, t)$ and $\vec{h}(x, t)$,
respectively, are approximated by

$$u^*(x, t; j, n) \equiv u^n_j + (u_x^n)_j (x - x_j) + (u_t^n)_j (t - t^n)$$  \quad (2.3)

and

$$\vec{h}^*(x, t; j, n) \equiv (au^*(x, t; j, n), u^*(x, t; j, n))$$  \quad (2.4)

Note that (i) $u^n_j$, $(u_x^n)_j$, and $(u_t^n)_j$
are constants in $SE(j, n)$, (ii) $(x_j, t^n)$ are the coor-
dinates of the mesh point $(j, n)$ with $x_j = jAx$ and $t^n = nAt$, and (iii)
Eq. (2.4) is the numerical analogue of the definition
$\vec{h} = (au, u)$.

Let $u = u^*(x, t; j, n)$ satisfy Eq. (2.1) within
$SE(j, n)$. Then one has $(u^n_j)_j = -a(u^n_j)_j$. As a result,
Eq. (2.3) reduces to

$$u^*(x, t; j, n) = u^n_j + (u_x^n)_j [(x - x_j) - a(t - t^n)]$$  \quad (2.5)

i.e., $u^n_j$ and $(u_x^n)_j$ are the only independent
marking variables associated with $(j, n)$.

Let $E_2$ be divided into nonoverlapping rectangular
regions (see Fig. 2(a)). As depicted in Figs. 2(c)–
2(e), (i) two such regions, i.e., $CE_-(j, n)$ and
$CE_+(j, n)$, are associated with each interior mesh
point $(j, n) \in \Omega$; and (ii) $CE(j, n)$ is the union of
$CE_-(j, n)$ and $CE_+(j, n)$.

Given the above preliminaries, we are ready to de-
scribe the existing CE/SE solvers for Eq. (2.1).

2.1. The $a$ scheme

Note that, among the line segments forming the
boundary of $CE_+(j, n)$, $\overline{AB}$ and $\overline{AD}$ belong to
$SE(j, n)$, while $\overline{CB}$ and $\overline{CD}$ belong to $SE(j - 1/2, n -
1/2)$. Similarly, the boundary of $CE_+(j, n)$ belongs
to either $SE(j, n)$ or $SE(j + 1/2, n - 1/2)$. As a
result, by imposing two conservation conditions at
each $(j, n) \in \Omega$, i.e.,

$$\oint_{S(CE_+(j, n))} \vec{h}^* \cdot d\vec{s} = 0, \quad \text{and}$$

$$\oint_{S(CE_-(j, n))} \vec{h}^* \cdot d\vec{s} = 0, \quad (j, n) \in \Omega$$  \quad (2.6)
and using Eqs. (2.4) and (2.5), one can obtain two equations for the two unknowns \( u^0_j \) and \( (u_x)^n_j \). In fact, let (i) \( \nu \overset{\text{def}}{=} a \Delta t / \Delta x \), and (ii) for any \((j, n) \in \Omega\),

\[
(u_x)^n_j \overset{\text{def}}{=} \frac{\Delta x}{4} (u_x)^n_j
\]

(2.7)

then Eq.(2.6) implies that (i)

\[
u^n_j = \frac{1}{2} \left\{ (1 + \nu) u^{n-1/2}_{j-1/2} + (1 - \nu) u^{n-1/2}_{j+1/2} \right. \\
\left. + (1 - \nu^2) \left( (u_x)^{n-1/2}_{j-1/2} - (u_x)^{n-1/2}_{j+1/2} \right) \right\} \quad (2.8)
\]

and, assuming \(|\nu| \neq 1\), (ii)

\[
(u_x)^n_j = (u_x^2)^n_j \quad (|\nu| \neq 1) \quad (2.9)
\]

with

\[
(u_x^2)^n_j \overset{\text{def}}{=} \frac{1}{2} \left\{ u^{n-1/2}_{j+1/2} - u^{n-1/2}_{j-1/2} - (1 + \nu)(u_x)^{n-1/2}_{j+1/2} - (1 - \nu)(u_x)^{n-1/2}_{j-1/2} \right\} \quad (|\nu| \neq 1) \quad (2.10)
\]

The \( a \) scheme, i.e., the inviscid case of the \( a-\mu \) scheme \([1,3,9]\), is formed by Eqs. (2.8) and (2.9). Note that, because

\[
\frac{\partial u}{\partial x} = \frac{\Delta x}{\Delta x} \frac{\partial u}{\partial x}
\]

if \( \tilde{x} \overset{\text{def}}{=} x / (4 \Delta x / 4) \), the normalized parameter \((u_x)^n_j\) can be interpreted as the value at \((j, n)\) of the derivative of \( u \) with respective to the normalized coordinate \( \tilde{x} \). Also note that the superscript symbol “a” in the parameter \((u_x^a)^n_j\) is introduced to remind the reader that Eq. (2.9) is valid for the \( a \) scheme.

The review of the \( a \) scheme is concluded with the following remarks:

(a) As shown in [3], the two amplification factors of the \( a \) scheme are identical to those of the leapfrog scheme. As a result, the \( a \) scheme is non-dissipative and it is stable if \(|\nu| < 1 \) (see the additional discussions given in Sec. 2.2).

(b) Note that derivation of Eqs. (2.8) and (2.9) can be facilitated by the following observations: because \( u^n(x, t; j, n) \) is linear in \( x \) and \( t \), it can be shown that the total flux of \( \tilde{h}^* \) leaving \( CE_-(j, n) \) or \( CE_+(j, n) \) through any of the four line segments that form its boundary is equal to the scalar product of the vector \( \tilde{h}^* \) evaluated at the midpoint of the line segment and the “surface” vector (i.e., the unit outward normal multiplied by the length) of the line segment.

(c) Because, for any \((j, n) \in \Omega\), the total flux of \( \tilde{h}^* \) leaving each of \( CE_-(j, n) \) and \( CE_+(j, n) \) vanishes (see Eq. (2.6)), \( CE_-(j, n) \) and \( CE_+(j, n) \), \((j, n) \in \Omega\), will be referred to as the conservation elements (CEs) of the \( a \) scheme. In addition, because the surface integration over any interface separating two neighboring CEs is evaluated using the information from a single CE, obviously the flux leaving one of these CEs through the interface is the negative of that leaving another CE through the same interface. As a result, the local conservation relations Eq. (2.6) lead to a global flux conservation relation, i.e., the total flux of \( \tilde{h}^* \) leaving the boundary of any space-time region that is the union of any combination of CEs will also vanish. In particular, because \( CE(j, n) \) is the union of \( CE_-(j, n) \) and \( CE_+(j, n) \),

\[
\int_{S(CE(j, n))} \tilde{h}^* \cdot \mathbf{d} \tilde{s} = 0 \quad (j, n) \in \Omega
\]

(2.11)

must follow from Eq. (2.6). In fact, it can be shown that Eq. (2.11) is equivalent to Eq. (2.8).

(d) In addition to the non-dissipative \( a \) scheme, as will be shown, there is a family of its dissipative extensions in which only the less stringent conservation condition Eq. (2.11) is assumed [3]. Because Eq. (2.11) is equivalent to Eq. (2.8), for each of these extensions, \((u_x)^n_j\) is still evaluated using Eq. (2.8) while \((u_x^a)^n_j\) is evaluated using an equation different from Eq. (2.9).

2.2. The \( a-\epsilon \) scheme and the \( c \) scheme

To proceed, consider any \((j, n) \in \Omega\). Then \((j \pm 1/2, n - 1/2) \in \Omega\). Let

\[
u^n_{j \pm 1/2} \overset{\text{def}}{=} u^n_{j \pm 1/2} + (\Delta t / 2)(u_x)^n_{j \pm 1/2} \quad (2.12)
\]

Substituting Eq. (2.7) and the relation \((u_x)^n_j = -a(u_x)^n_j\) into Eq. (2.12) and using the definition \( \nu = a \Delta t / \Delta x \), one has

\[
u^n_{j \pm 1/2} = (u - 2 \nu u_x^n)_{j \pm 1/2} \quad (2.13)
\]

Note that, to simplify notation, in the above and hereafter we adopt a convention that can be explained using the expression on the right side of Eq. (2.13) as an example, i.e.,

\[
(u - 2 \nu u_x^n)_{j \pm 1/2} = u^n_{j \pm 1/2} - 2 \nu (u_x)_{j \pm 1/2} \quad (2.14)
\]

Also note that, by definition, \((j \pm 1/2, n) \notin \Omega\) if \((j, n) \in \Omega\). Thus \((u_x^a)^n_j\) is associated with a mesh
Eq. (2.17) reduces to Eq. (2.18). As it turns out, implementation of a multidimensional Euler version of the \( e \) scheme does not require inverting any system of equations while a similar implementation involving a version of any other \( a-\epsilon \) scheme \((\epsilon \neq 1/2)\) generally requires inverting, per mesh point and per time step, a system of several linear equations (to be exact, a system of eight and fifteen equations, respectively, for 2D and 3D Euler equations). As such, it is much more cost effective to use a multidimensional Euler version of the \( e \) scheme than using that of any other \( a-\epsilon \) scheme. Partly for this reason, extensions of the \( e \) scheme have been used extensively.

(b) For the \( a-\epsilon \) scheme, it is shown in [3] that the principal and spurious amplification factors per \( \Delta t \), respectively, are \((\lambda_+)^2\) and \((\lambda_-)^2\) with

\[
\lambda_{\pm}(\epsilon, \nu, \theta) \stackrel{\text{def}}{=} \epsilon \cos(\theta/2) - i\nu \sin(\theta/2) \pm \sqrt{(1-\epsilon) [(1-\epsilon)\cos^2(\theta/2) + (1-\nu^2)\sin^2(\theta/2)]]}.
\]

Here (i) \( \nu \) is \( \equiv \sqrt{-1} \), and (ii) \( \theta, -\pi < \theta \leq \pi \), is the phase angle variation per \( \Delta x \). In addition, it is shown that (i) the necessary and sufficient conditions for the stability of the \( a-\epsilon \) scheme are

\[
0 \leq \epsilon \leq 1, \quad \text{and} \quad |\nu| < 1
\]

and (ii) the \( a-\epsilon \) scheme becomes progressively diffusive as the value of \( \epsilon \) increases from 0 to 1. Note that, unless specified otherwise, in the remainder of the paper the ranges of \( \epsilon, \nu \) and \( \theta \), respectively, are defined by Eq. (2.20) and \(-\pi < \theta \leq \pi\).

(c) For the plane wave solution, \( u = e^{ik(x-\nu t)} \),

the exact amplification factor per \( \Delta t \)

\[
\frac{e^{ik[x-a(t+\Delta t)]}}{e^{ik(x-\nu t)}} = e^{-ik\Delta t} = e^{-i\nu \theta}
\]

where \( \theta = k \Delta x \).

(d) According to Eq. (2.19), \([\lambda_\pm(0,\nu,\theta)]^2\), the amplification factors of the \( e \) scheme (which corresponds to the case \( \epsilon = 0 \)), have the following properties:

\[
[|\lambda_\pm(0,\nu,\theta)|^2] = 1
\]

\[
\lim_{\nu \to \pm 1} [\lambda_+(0,\nu,\theta)]^2 = e^{\mp i\theta}
\]

\[
\lim_{\nu \to \pm 1} [\lambda_-(0,\nu,\theta)]^2 = e^{\pm i\theta}
\]
and
\[ [\lambda_\pm(0, 0, \theta)]^2 = 1 \] (2.25)

On the other hand, \( e^{-i\nu \theta} \), the exact amplification factor, has the following properties:
\[ [e^{-i\nu \theta}] = 1 \] (2.26)
\[ \lim_{\nu \to \pm 1} e^{-i\nu \theta} = e^{\mp i \theta} \] (2.27)
and
\[ e^{-i\nu \theta} = 1 \quad \text{if} \quad \nu = 0 \] (2.28)

For the \( a \) scheme, Eqs. (2.22)–(2.28) imply that: (i) the two amplification factor of the scheme, and the exact amplification factor all have the same constant absolute value (\( = 1 \)) and, thus, the scheme is non-dissipative; (ii) in the limit of \( \nu \to 1 \) (i.e., \( \nu \to 1 \) or \( \nu \to -1 \)), the principal amplification factor is identical to the exact amplification factor and, thus, the former has no dissipative or dispersive error in this limit; (iii) also in the limit of \( \nu \to 1 \), the spurious amplification factor is exactly the negative of that associated with the exact amplification factor and, thus, the spurious amplification factor has a large dispersive error in this limit except when \( |\theta| \ll 1 \) (i.e., when the wavelengths of the errors \( \gg 1 \)); and (iv) when \( \nu = 0 \), the two amplification factors of the scheme, and the exact amplification factor are all equal to 1 and, thus, the two amplification factors of the scheme have no dissipative or dispersive error if \( \nu = 0 \). Because the accuracy of a scheme is essentially hinged on the behaviors of the principal amplification factor \([1]\), according to the facts stated above, the \( a \) scheme tends to become very accurate when \( |\nu| \) approaches 1 or 0. However, the short-wavelength errors associated with the spurious amplification factor (which could be introduced at \( t = 0 \) as a result of an inaccurate initial-value specification \([1]\)) may appear in a solution as persistent (i.e., non-dissipative) numerical wiggles when \( |\nu| \) approaches 1 \([19]\).

(e) According to Eq. (2.19), \([\lambda_\pm(1/2, \nu, \theta)]^2\), the amplification factors of the \( c \) scheme (which corresponds to the case \( \epsilon = 1/2 \)), have the following properties:
\[ \lim_{\nu \to \pm 1} [\lambda_+(1/2, \nu, \theta)]^2 = e^{\mp i \theta} \] (2.29)
\[ \lim_{\nu \to \pm 1} [\lambda_-(1/2, \nu, \theta)]^2 = -\sin^2(\theta/2) \] (2.30)
and
\[ [\lambda_\pm(1/2, 0, \theta)]^2 = \frac{1}{2} \left[ 1 \pm \cos(\theta/2) \sqrt{2 - \cos^2(\theta/2)} \right] \] (2.31)

For the \( c \) scheme, Eqs. (2.27)–(2.31) imply that: (i) in the limit of \( |\nu| \to 1 \), the principal amplification factor is identical to the exact amplification factor and, thus, the former has no dissipative or dispersive errors in this limit; (ii) also in the limit of \( |\nu| \to 1 \), the spurious amplification factor has large dissipative and dispersive errors; and (iii) when \( \nu = 0 \), the two amplification factors of the scheme generally have large dissipative errors but no dispersive errors. According to the facts stated above, like the \( a \) scheme, the \( c \) scheme also tends to become very accurate when \( |\nu| \) approaches 1. However, unlike the \( a \) scheme, the errors associated with the spurious amplification factor of the \( c \) scheme generally do die out rapidly when \( |\nu| \) approaches 1. Also, in sharp contrast to the \( a \) scheme, the \( c \) scheme becomes highly dissipative when \( \nu \) approaches 0.

In Sec. 3, it will be shown that new solvers of Eq. (2.1) can indeed be constructed such that they possess all the advantages but none of the disadvantages listed above. Specifically, each of these solvers will be formed by Eq. (2.8) and a new equation in which \((u^c)_j^n\) is evaluated using a simple central-differencing procedure similar to that used to obtain \((u^e)_j^n\). In addition, \((u^c)_j^n\) so obtained will be (i) identical to \((u^e)_j^n\) in the limit of \( |\nu| \to 1 \), and (ii) identical to \((u^e)_j^n\) when \( \nu = 0 \). As such, each of these new solvers (i) is comparable to the \( c \) scheme in ease of implementation; (ii) becomes the \( c \) scheme in the limit of \( |\nu| \to 1 \); and (iii) becomes the \( a \) scheme when \( \nu = 0 \).

2.3. A special wiggle-suppressing scheme

If discontinuities are present in a numerical solution, any \( a-\epsilon \) scheme such as the \( c \) scheme is not equipped to suppress numerical wiggles that generally appear near these discontinuities. To serve as a preliminary for future development, here we shall briefly review an extension of the \( c \) scheme which was introduced as a remedy for this deficiency \([3,35]\).

To proceed, let
\[(u^c_\pm)_j^n \overset{\text{def}}{=} \frac{1}{2} (u_\pm^n - u_{\pm-1/2}^n) = \frac{\Delta x}{4} \left( \frac{u_j^n - u_{j-1/2}^n}{\Delta x/2} \right) \] (2.32)
and

\[
(u_{x+}^n)_j \overset{\text{def}}{=} \frac{1}{2} (u_{j+1/2}^n - u_j^n) = \frac{\Delta x}{4} \left( \frac{u_{j+1}^n - u_j^n}{\Delta x/2} \right)
\]

(2.33)

e.g., \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\) are two (normalized) numerical analogues of \(\partial u/\partial x\) at \((j, n)\) with one being evaluated from the left and another from the right. It can be shown that

\[
(u_{x-}^n)_j^n = \frac{1}{2} (u_{x-}^n + u_{x+}^n)_j^n
\]

(2.34)

e.g., \((u_{x-}^n)_j^n\) is the simple average of \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\). As such, the \(c\) scheme can be extended by replacing \((u_{x-}^n)_j^n\) in Eq. (2.18) with an weighted average of \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\). In other words, the resulting extension is formed by Eq. (2.8) and

\[
(u_{x}^n)_j^n = (w_{-})_j^n (u_{x-}^n)_j^n + (w_{+})_j^n (u_{x+}^n)_j^n
\]

(2.35)

where \((w_{-})_j^n\) and \((w_{+})_j^n\), the weight factors associated with \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\) respectively, must satisfy the condition

\[
(w_{-})_j^n + (w_{+})_j^n = 1
\]

(2.36)
at all \((j, n) \in \Omega\). In addition, the expression on the right side of Eq. (2.35) represents an interpolation (rather than an extrapolation) of \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\) if and only if

\[
(w_{-})_j^n \geq 0 \quad \text{and} \quad (w_{+})_j^n \geq 0
\]

(2.37)

Let \(x_{-}, x_{+}\), and \(\alpha \geq 0\) be real variables. Let \(W_{-}\) and \(W_{+}\) be the functions defined by: (i)\n
\[
W_-(x_{-}, x_{+}; \alpha) = W_+(x_{-}, x_{+}; \alpha) = 1/2 \text{ if } x_{-} = x_{+} = 0; \text{ and (ii)}
\]

\[
W_-(x_{-}, x_{+}; \alpha) = \frac{|x_+|}{|x_-|^\alpha + |x_+|^\alpha} (|x_-| + |x_+| > 0)
\]

(2.38a)

and

\[
W_+(x_{-}, x_{+}; \alpha) = \frac{|x_-|}{|x_-|^\alpha + |x_+|^\alpha} (|x_-| + |x_+| > 0)
\]

(2.38b)

if either \(x_{-} \neq 0\) or \(x_{+} \neq 0\). Furthermore, let

\[
(w_{\pm})_j^n = W_\pm ((u_{x-}^n)_j^n, (u_{x+}^n)_j^n, \alpha)
\]

(2.39)

Then \((w_{-})_j^n\) and \((w_{+})_j^n\) so defined satisfy Eqs. (2.36) and (2.37) and have the property that

\[
(w_{-})_j^n = (w_{+})_j^n = 1/2 \text{ if } \alpha = 0
\]

or \|[u_{x-}^n]_j^n\| = |[u_{x+}^n]_j^n| \]

(2.40)

Note that: (i) to avoid dividing by zero, in practice a small positive number such as \(10^{-20}\) is added to each of the denominators in Eqs. (2.38a,b); and (ii) the special cases of Eqs. (2.38a,b) with \(\alpha = 1\) and \(\alpha = 2\) are used in the slope-limiter proposed by van Leer [38], and van Albada et al. [39].

An extension of the \(c\) scheme is formed by Eqs. (2.8) and (2.35) with \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\) being defined by Eq. (2.39). Let \(\alpha > 0\) and \([u_{x-}^n]_j^n \neq [u_{x+}^n]_j^n\). Then Eqs. (2.38a,b) imply that, of \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\), the one with smaller absolute value is associated with an weight factor \(> 1/2\). This observation coupled with Eqs. (2.34)–(2.37) leads to the conclusion that, of \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\), \((u_{x-}^n)_j^n\) will have an algebraic value closer to the one with smaller absolute value if \((u_{x-}^n)_j^n\) is evaluated as a weighted average of \((u_{x-}^n)_j^n\) and \((u_{x+}^n)_j^n\) according to Eq. (2.35). As a result, \((u_{x-}^n)_j^n\) so evaluated has a smaller absolute value than that evaluated using Eq. (2.18). In turn, numerical wiggles or overshoots can be annihilated by the additional numerical dissipation introduced as a result of this local “flattening” of \((u_{x-}^n)_j^n\). It has been shown numerically that the extension is stable if \(|\nu| < 1\) and \(\alpha \geq 0\). Moreover, as a result of Eqs. (2.18), (2.34), (2.35) and (2.40), (i) the extension reduces to the \(c\) scheme when \(\alpha = 0\); and (ii) even if \(\alpha > 0\), the extension behaves very much like the \(c\) scheme in any smooth solution region (where the condition \((u_{x-}^n)_j^n = (u_{x+}^n)_j^n\) more or less prevails) or at a solution extremum (where the condition \((u_{x-}^n)_j^n = -(u_{x+}^n)_j^n\) more or less prevails). As such, the wiggle-suppressing power of the extension takes effect only if \(\alpha > 0\) and only in a solution region where \([u_{x-}^n]_j^n\) and \([u_{x+}^n]_j^n\) differ substantially.

3. The \(c-x\) and \(c-x^2\) schemes

In this section, the ideal solvers of Eq. (2.1) mentioned at the end of Sec. 2.2 will be constructed. As a preliminary, we shall show that \((u_{x-}^n)_j^n\) can also be cast into a central-difference form when \(\nu = 0\).

To proceed, note that by assumption \(a \neq 0\) Thus \(\nu = 0\) if and only if \(\Delta t = 0\). Because \(|EF| = |AB| = |CD| = 0\) (see Fig. 2(c,d)) when \(\Delta t = 0\), the two conservation conditions given in Eq. (2.6), for the case \(\nu = 0\), respectively reduce to: (i) the flux leaving \(CE_{+}(j, n) \) through the top face \(\overline{AE}\) is equal to that entering the same CE through the bottom face \(\overline{ED}\); and (ii) the flux leaving \(CE_{+}(j, n) \) through the top face \(\overline{AB}\) is equal to that entering the same CE through the bottom face \(\overline{CD}\). According to Remark (b) given at the end of Sec. 2.1, the flux leaving \(CE_{+}(j, n) \) through the top face \(\overline{AB}\) is equal to the value of \(u^*\) at the midpoint of \(\overline{AB}\) (evaluated using the marching variables at point \(A\)) multiplied by \(|\overline{AB}|\), while that entering it through the bottom face \(\overline{CD}\) is equal to the value of \(u^*\) at the midpoint.
of $\overline{ED}$ (evaluated using the marching variables at point $E$) multiplied by $[\overline{ED}]$. With the aid of these observations and the fact that $[\overline{AF}] = [\overline{ED}]$, the above condition (i) implies that, when $\nu = 0$, the value of $u^*$ at the midpoint of $\overline{AF}$ evaluated using the marching variables at point $A$ is equal to that at the midpoint of $\overline{ED}$ evaluated using the marching variables at point $E$. As such, the first conservation condition given in Eq. (2.6) is equivalent to

$$(u + u_x)^n_j = (u + u_x)^{n-1/2}_{j+1/2} \quad (\nu = 0) \quad (3.1)$$

if $\nu = 0$. Similarly, by using the above condition (ii), it can be shown that the second conservation condition given in Eq. (2.6) is equivalent to

$$(u - u_x)^n_j = (u - u_x)^{n-1/2}_{j-1/2} \quad (\nu = 0) \quad (3.2)$$

if $\nu = 0$. Because Eqs. (2.8) and (2.9) (which form the $\alpha$ scheme) are equivalent to Eqs. (2.6) if $|\nu| \neq 1$, they must be equivalent to Eq. (3.1) and (3.2) when $\nu = 0$. In fact, by subtracting Eq. (3.2) from Eq. (3.1), one obtains Eq. (2.9) where $(u_x^2)^n_j$ is the reduced form of Eq. (2.10) for the case $\nu = 0$, i.e.,

$$(u_x^2)^n_j = \frac{1}{2} \left[ (u - u_x)^{n-1/2}_{j+1/2} - (u + u_x)^{n-1/2}_{j-1/2} \right] \quad (\nu = 0) \quad (3.3)$$

Similarly, by summing Eqs. (3.1) and (3.2), one can obtain the reduced form of Eq. (2.8) for the case $\nu = 0$.

With the aid of Eq. (3.3) and the facts that: (i) $(u - u_x)^{n-1/2}_{j+1/2}$ and $(u + u_x)^{n-1/2}_{j-1/2}$, respectively, represent an approximation of $u$ at the midpoint of $\overline{ED}$ and that at the midpoint of $\overline{CD}$ (see Fig. 2.c.d.i); and (ii) the distance between the two midpoints referred to above is $\Delta x/2$, it becomes obvious that, for the special case $\nu = 0$, $(u_x^2)^n_j$ is indeed a central-difference approximation of $\partial u / \partial x$ at $(j, n-1/2)$ (which is identical to $(j, n)$ when $\nu = 0$), normalized by the factor $\Delta x/4$. QED.

According to the above discussions, construction of the ideal solvers defined at the end of Sec. 2.2 is hingeed on finding central-difference approximations for $(u_x^2)^n_j$ such that each approximation (i) becomes $(u_x^2)^n_j$ in the limit of $|\nu| \to 1$, and (ii) reduces to the expression on the right side of Eq. (3.3) when $\nu = 0$. As a result of these observations, these new solvers can easily be constructed as the subschemes of the c-τ scheme, a new solver of Eq. (2.1) to be described immediately.

3.1. The c-τ scheme

To proceed, refer to Fig. 3. Here $M_+$ and $M_-$ denote the midpoints of $\overline{AF}$ and $\overline{AB}$, respectively. Also $P_+$ and $P_-$ are two points on $\overline{BF}$ that satisfy the following two conditions: (i) $P_+$ is to the right (left) of $M_+$ if and only if $P_-$ is to the left (right) of $M_-$; and (ii) $|M_+ P_+| = |M_- P_-|$, i.e., $M_+ P_+$ and $M_- P_-$ have the same length. In addition, let the parameter $\tau$ be defined by: (i) $\tau \Delta x/4 = [M_+ P_+]$ if $P_+$ is to the right of $M_+$; and (ii) $\tau \Delta x/4 = [M_- P_-]$ if $P_-$ is to the left of $M_-$. Obviously, it follows from the above definitions that (i) $\tau \Delta x/4 = [M_+ P_+]$ if $P_+$ is to the left of $M_+$; and (ii) $\tau \Delta x/4 = [M_- P_-]$ if $P_-$ is to the right of $M_-$. Moreover, let

$$u'(P_+) \overset{\text{def}}{=} u + (\Delta t/2)u_x - (1 - \tau)(\Delta x/4)u_x^{n-1/2} \quad (3.4)$$

$$u'(P_-) \overset{\text{def}}{=} u + (\Delta t/2)u_x + (1 - \tau)(\Delta x/4)u_x^{n-1/2} \quad (3.5)$$

By definition, $u'(P_+)$ is a first-order Taylor’s approximation of $u$ at $P_+$ evaluated using the marching variables at point $E$, while $u'(P_-)$ is a first-order Taylor’s approximation of $u$ at $P_-$ evaluated using the marching variables at point $C$. Also note that, by using Eq. (2.7) and the relation $(u_x)^n_j = -a(u_x)^n_j$, Eqs. (3.4) and (3.5) can be simplified as

$$u'(P_+) = [u - (1 + 2\nu - \tau)u_x]^{n-1/2}_{j+1/2} \quad (3.6)$$

and

$$u'(P_-) = [u + (1 - 2\nu - \tau)u_x]^{n-1/2}_{j-1/2} \quad (3.7)$$

respectively.

At this juncture, note that $P_+$ and $P_-$ generally lie outside of $\text{SE}(j + 1/2, n - 1/2)$ and $\text{SE}(j - 1/2, n - 1/2)$, respectively. Yet here, by definition, $u'(P_+)$ and $u'(P_-)$ are evaluated as though $P_+$ and $P_-$ lie within $(j + 1/2, n - 1/2)$ and $(j - 1/2, n - 1/2)$, respectively. At first glance, the current practice is inconsistent with a previously established rule. However, as explained by the reasons given below, the definition of $u'(P_+)$ and $u'(P_-)$ is perfectly legitimate:

(a) Recall that solution elements were introduced such that the boundary of a CBE can be divided into several component parts with each of them belonging to a unique solution element. As such, the flux over a component part that belongs to a special solution element, say $\text{SE}(j, n)$, can be unambiguously determined in terms of the marching variables at the mesh point $(j, n)$. In other words, in related to evaluation of any flux conservation condition over any CBE, Eqs. (2.3)–(2.5) can be applied only to a point $(x, t) \in \text{SE}(j, n)$.

(b) On the other hand, $u'(P_+)$ and $u'(P_-)$ introduced here have nothing to do with flux evalua-
tion. In fact, they will be used only in the construction of some numerical analogues of \( \partial u / \partial x \) at \((j, n)\).

To proceed, note that: (i) \( A \) (i.e., the mesh point \((j, n)\)) is the midpoint of \( P_+ - P_- \), and (ii) \( |P_- - P_+| = (1 + \tau) \Delta x / 2 \). Thus

\[
(\hat{u}_x)_j^n = \left[ u\left(\frac{P_+ - u'(P_-)}{2(1 + \tau)} \right) \right]
\]

represents a central-difference approximation of \( \partial u / \partial x \) at the mesh point \((j, n)\), normalized by the factor \( \Delta x / 4 \). Thus the new scheme formed by Eqs. (2.8) and

\[
(\hat{u}_x)_j^n = (u_x)_j^n \quad (3.9)
\]

represents a solver for Eq. (2.1). Because (i) \((\hat{u}_x)_j^n\) represents a central-difference approximation of \((u_x)_j^n\), and (ii) the approximation is associated with the parameter \( \tau \), hereafter the new scheme will be referred to as the c-\( \tau \) scheme.

To explore the c-\( \tau \) scheme, note that Eqs. (3.6)–(3.8) can be combined to yield

\[
(\hat{u}_x)_j^n = \frac{1}{2(1 + \tau)} \left\{ [u - (1 + 2\nu - \tau)u_x]^{n+1/2} - [u + (1 - 2\nu - \tau)u_x]^{n-1/2} \right\} \quad (\tau \neq -1) \quad (3.10)
\]

Moreover, by using Eqs. (2.10), (2.16) and (3.10), one has

\[
(\hat{u}_x)_j^n = (u_x)_j^n + \frac{2\tau}{1 + \tau} (u_x^n - u_x^n) + \frac{\nu(1 - \tau)}{2(1 + \tau)} \left[ (u_x)^{n+1/2} - (u_x)^{n+1/2} - (u_x)_{j+1/2}^{n-1/2} \right] \quad (3.11)
\]

By comparing Eq. (3.11) with (2.17), one concludes that the c-\( \tau \) scheme is generally different from the a-\( \epsilon \) scheme. In fact, a special case of the c-\( \tau \) scheme can be turned into that of the a-\( \epsilon \) scheme and vice versa if and only if either (i) \( \tau = 1 \) or (ii) \( \nu = 0 \). For the case \( \tau = 1 \), Eq. (3.11) implies that \((\hat{u}_x)_j^n = (u_x)_j^n\). In other words, the c-\( \tau \) scheme is the special case of the c-\( \tau \) scheme with \( \tau = 1 \), a fact that can also be deduced from the observation that the points \( P_+ \) and \( P_- \) depicted in Fig. 3, respectively, coincide with points F and B (i.e., the mesh points \((j + 1/2, n)\) and \((j - 1/2, n)\)) if \( \tau = 1 \). On the other hand, it is seen that, when \( \nu = 0 \), the c-\( \tau \) scheme become the a-\( \epsilon \) scheme with \( \epsilon = \tau/(1 + \tau) \). In fact one can further deduce that c-\( \tau \) scheme reduces to the a scheme if and only if \( \nu = \tau = 0 \).

Furthermore, by studying the amplification matrix and the amplification factors of the c-\( \tau \) scheme [40], it has been established that:

(a) The c-\( \tau \) scheme is stable if

\[
\tau \geq \tau_0(|\nu|), \quad \text{and} \quad |\nu| < 1 \quad (3.12)
\]

where \( \tau_0(s) \) \((0 \leq s < 1) \) is a numerically established function. As shown in Fig. 4, \( \tau(s) \) is a strictly increasing function of \( s \) with the following properties:

\[
\tau(0) = 0; \quad \lim_{s \to 1-} \tau(s) = 1; \quad \text{and} \quad 0 < \tau(s) < s \quad \text{if} \quad 0 < s < 1 \quad (3.13)
\]

(b) The c-\( \tau \) scheme is also stable if \( \tau > 1 \) and \(|\nu| \leq 1 \). However, it is unstable if either (i) \(|\nu| > 1 \) or (ii) \(|\nu| < 1 \) and \( \tau < \tau_0(|\nu|) \).

(c) For any given fixed value of \(|\nu| < 1 \), the c-\( \tau \) scheme generally becomes more dissipative as the value of \( \tau \) increases from its minimum \( \tau_0(|\nu|) \). With the above preliminaries, the ideal solvers of Eq. (2.1) defined at the end of Sec. 2.2 will be constructed in Sec. 3.2.

3.2. The c-\( \tau^* \) scheme

The value of \( \tau \) used in the c-\( \tau \) scheme generally can be chosen independent of \( \nu \). Here we consider a subset of the c-\( \tau \) scheme in which \( \tau \) is a function of \(|\nu| \). Specifically, for each member in this subset, (i) there exists a strictly monotonically increasing smooth function \( h(s) \), \( 0 \leq s < 1 \), which has the following properties:

\[
h(0) = 0; \quad \lim_{s \to 1-} h(s) = 1; \quad \text{and} \quad h(s) \geq \tau_0(s) \quad \text{if} \quad 0 < s < 1 \quad (3.14)
\]

and

\[
(\nu) = h(|\nu|) \quad (|\nu| < 1) \quad (3.15)
\]

Note that, using the definition of \( h \) and Eq. (3.15), one can easily infer from Fig. 3 a simple relation between the value of \(|\nu| \) and the locations of \( P_+ \) and \( P_- \), i.e., as the value of \(|\nu| \) increases from 0 to 1, \( P_+ \) will move away from \( M_+ \) and edge toward the mesh point \((j + 1/2, n)\) while \( P_- \) will move away from point \( M_- \) and edge toward the mesh point \((j - 1/2, n)\). Also note that, by using Eqs. (3.13)–(3.15), one can show that:

\[
\tau = 0 \quad \text{if} \quad \nu = 0 \quad (3.16)
\]
and (iii)

\[ \tau \geq \tau_0(|\nu|) \quad (|\nu| < 1) \quad (3.18) \]

Recall that (i) \((\hat{u}_\tau)_{j}^n = (u_\tau^+)^n\) if \(\tau = \nu = 0\); and (ii) \((\hat{u}_\tau)_{j}^n = (u_\tau^-)^n\) if \(\tau = 1\). As such, Eqs. (3.16) and (3.17) imply that, for each member in the subset, (i) \((u_\tau)_{j}^n = (u_\tau^+)^n\) if \(\nu = 0\); and (ii) \((u_\tau)_{j}^n = (u_\tau^-)^n\) in the limit of \(|\nu| \to 1\). In other words, all members in the subset are ideal solvers in the domain \(|\nu| < 1\). Moreover, by using Eq. (3.12) and (3.18), one can also show that these ideal solvers are also stable in the same domain.

Corresponding to infinitely many choices of \(h\) that meet all the requirements stated earlier, there are infinitely many members in the subset. One special member of this subset with

\[ h(s) = s \quad (0 \leq s < 1) \quad (3.19) \]

is explored in simply as the \(c-\tau^*\) scheme. Also, unless specified otherwise, in the remainder of this paper Eq. (3.15) will be assumed for a given function \(h\).

4. Extensions of the \(c-\tau^*\) scheme and related weighted averagings

To proceed, let

\[ (\hat{u}_{\tau^-})_{j}^n \equiv \frac{u_j^n - u'(P_-)}{1 + \tau} = \frac{\Delta x}{4} \left( \frac{u_j^n - u'(P_-)}{(1 + \tau)\Delta x/4} \right) \quad (4.1) \]

and

\[ (\hat{u}_{\tau^+})_{j}^n \equiv \frac{u'(P_+)^n - u_j^n}{1 + \tau} = \frac{\Delta x}{4} \left( \frac{u'(P_+)^n - u_j^n}{(1 + \tau)\Delta x/4} \right) \quad (4.2) \]

Because \(\left[ \frac{\Delta P_-}{\Delta P_+} \right] = \left[ \frac{\Delta P_-}{\Delta P_+} \right] = (1 + \tau)\Delta x/4 \) (see Fig. 3), it is easy to see that \((\hat{u}_{\tau^-})_{j}^n\) and \((\hat{u}_{\tau^+})_{j}^n\) are two normalized one-sided difference approximations of \(\partial u/\partial x\) at the mesh point \((j, n)\) with one being evaluated from the left and another from the right. Also, it follows immediately from Eqs. (3.8), (4.1) and (4.2) that

\[ (u_\tau)_{j}^n = \frac{1}{2} \left( (\hat{u}_{\tau^-})_{j}^n + (\hat{u}_{\tau^+})_{j}^n \right) \quad (4.3) \]

Moreover, by (i) substituting Eqs. (2.8), (3.6) and (3.7) into Eqs. (4.1) and (4.2), and (ii) using Eqs. (3.10) and (3.16), one arrives at the conclusion that

\[ (u_{\tau^-})_{j}^n = (\hat{u}_{\tau^+})_{j}^n = (\hat{u}_{\tau^-})_{j}^n \quad (\nu = 0) \quad (4.4) \]

when \(\nu = 0\).

With the above preliminaries, several extensions of the \(c-\tau^*\) scheme will be constructed in the following subsections.

4.1. Scheme w-1

A comparison of Eqs. (4.1)-(4.3) with Eqs. (2.32)-(2.34) reveals that an obvious extension of the \(c-\tau^*\) scheme can be obtained by replacing \((u_{\tau^-})_{j}^n\) and \((u_{\tau^+})_{j}^n\) in Eqs. (2.35) and (2.39) with \((\hat{u}_{\tau^-})_{j}^n\) and \((\hat{u}_{\tau^+})_{j}^n\), respectively. In other words, the new extension is formed by Eq. (3.8) and

\[ (u_\tau)_{j}^n = (w_-)_{j}^n((\hat{u}_{\tau^-})_{j}^n + (w_+)^n(\hat{u}_{\tau^+})_{j}^n) \quad (4.5) \]

with

\[ (w_\pm)_{j}^n = W_\pm ((\hat{u}_{\tau^-})_{j}^n, (\hat{u}_{\tau^+})_{j}^n, \alpha) \quad (4.6) \]

Because the scheme is the first extension of the \(c-\tau^*\) scheme in which \((u_{\tau^-})_{j}^n\) is expressed as an weighted average of \((\hat{u}_{\tau^-})_{j}^n\) and \((\hat{u}_{\tau^+})_{j}^n\), for simplicity, hereafter it will be referred to as Scheme w-1. It has been shown numerically that Scheme w-1 is stable if \(|\nu| < 1\) and \(\alpha \geq 0\).

Note that, as a result of Eqs. (2.38a,b), (4.4) and (4.6), one concludes that, for any given \(\alpha \geq 0\), \((w_-)_{j}^n = (w_+)^n = 1/2\) if \(\nu = 0\). In other words, for Scheme w-1, the “weighted” average on the right side of Eq. (4.5) becomes a simple average if \(\nu = 0\). According to an explanation given in the last paragraph of Sec. 2, this implies that Scheme w-1 will lose its capability to suppress wiggles or overshoots when \(\nu\) becomes small. For this reason, even though the Euler version of Scheme w-1 performs much better than that of the special scheme referred to in Sec. 2 in its ability to resolve shocks and contact discontinuities crisply in a wide range (from 1 to less than 0.001) of the global CFL number (i.e., the maximal value of local CFL numbers), it has a serious shortcoming, i.e., wiggles or overshoots can appear near a discontinuity in a generated solution when the local CFL number there becomes extremely small. In the following, it will be shown that this weakness can be overcome by simple modifications of Eq. (4.6).

4.2. Scheme w-2

A new scheme, referred to as Scheme w-2 is formed by Eqs. (3.8) and (4.5) with \((w_\pm)_{j}^n\) being given by Eq. (2.39). In other words, although \((w_\pm)_{j}^n\) is still constructed as an weighted average of \((\hat{u}_{\tau^-})_{j}^n\) and \((\hat{u}_{\tau^+})_{j}^n\), the associated weight factors \((w_\pm)_{j}^n\) are evaluated using \((u_{\tau^-})_{j}^n\) and \((u_{\tau^+})_{j}^n\). Because the last two parameters, respectively, are identical to the special cases of \((\hat{u}_{\tau^-})_{j}^n\) and \((\hat{u}_{\tau^+})_{j}^n\) with \(\tau = 1\) (see Eqs. (2.12), (2.32), (2.33), (3.4), (3.5), (4.1), and (4.2)), their values do not vary with \(\nu\). As such, \((w_\pm)_{j}^n \neq 1/2\) and therefore the weighted average on
the right side of Eq. (4.5) will not turn into a simple average when \( \nu = 0 \). In other words, Scheme w-2 is still capable of annihilating the numerical wiggles near a discontinuity even if \( \nu \) becomes small. It has been shown numerically that Scheme w-2 again is stable if \(|\nu| < 1 \) and \( \alpha \geq 0 \).

Note that a possible drawback of Scheme w-2 is that the relation \( ||(u_{\sigma_-})^n_j|| < ||(u_{\sigma+})^n_j|| \) (\( ||(u_{\sigma-})^n_j|| > ||(u_{\sigma+})^n_j|| \)) does not automatically follow from \( ||(\bar{u}_{\sigma-})^n_j|| < ||(\bar{u}_{\sigma+})^n_j|| \) (\( ||(\bar{u}_{\sigma-})^n_j|| > ||(\bar{u}_{\sigma+})^n_j|| \)) and vice versa. As a result, at some local mesh points, it may happen that, of \((\bar{u}_{\sigma-})^n_j\) and \((\bar{u}_{\sigma+})^n_j\), the one with smaller absolute value may not be associated with a weight factor \( > 1/2 \). According to a discussion given in the last paragraph of Sec. 2, this implies that there is no guarantee that, at all localities, the weighted-averaging induced numerical dissipation will be available to suppress wiggles or overshoots. Despite this possible failure, fortunately it has been demonstrated numerically that, not only are they capable of suppressing wiggles or overshoots robustly, Scheme w-2 and its Euler extensions are also highly accurate.

In the following, schemes that overcome the weakness of Scheme w-1 and also avoid the theoretically possible failing associated with Scheme w-2 will be constructed using new weighted-averaging formulae more advanced than that given in Eqs. (2.38a,b).

4.3. New weighted-averaging techniques

To pave the way, first we shall discuss a limitation of Eqs. (2.38a,b) as a generator of weight factors.

Let \( x_\pm \neq 0 \). Then, for a given \( \alpha \geq 0 \), obviously \( W_- \to 1/2 \) and \( W_+ \to 1/2 \) as \( |x_+/x_-| \to 1 \). As such, when \( |x_+/x_-| \) is very close to 1, then both \( W_- \) and \( W_+ \) will be very close to \( 1/2 \) unless \( \alpha \gg 1 \). As a result, in case that (i) \((\bar{u}_{\sigma\pm})^n_j \neq 0 \), (ii) \((\bar{u}_{\sigma\pm})^n_j)/(\bar{u}_{\sigma\mp})^n_j\) is very close to 1; and (iii) Eqs. (4.6) is assumed, then the only way to prevent the weighted average that appears on the right side of Eq. (4.5) from becoming almost a simple average is to increase the value of \( \alpha \) used. However, this approach may be impracticable because numerical evaluation of a quantity such as \( x^\alpha \) for any real number \( x \) generally is hampered by round-off errors and thus becomes highly inaccurate if the value of \( \alpha \) becomes too large, say 100. It is the purpose of this subsection to introduce new weighted-averaging techniques that do not have the limitation discussed above.

For motivation, note that Eqs. (4.5) and (4.6) can be expressed as

\[
(u_{\sigma\pm})^n_j = w_1 x_1 + w_2 x_2
\]

and

\[
w_1 = \frac{s_1}{s_1 + s_2} \quad \text{and} \quad w_2 = \frac{s_2}{s_1 + s_2} \quad (s_1 + s_2 > 0)
\]

respectively if

\[
x_1^\text{def} = (\bar{u}_{\sigma-})^n_j \quad \text{and} \quad x_2^\text{def} = (\bar{u}_{\sigma+})^n_j
\]

\[
w_1^\text{def} = (w_-)^n_j \quad \text{and} \quad w_2^\text{def} = (w_+)^n_j
\]

\[
s_1^\text{def} = ||(\bar{u}_{\sigma-})^n_j|| \quad \text{and} \quad s_2^\text{def} = ||(\bar{u}_{\sigma+})^n_j|| \quad (\alpha \geq 0)
\]

Equation (4.7) represents an average weighted average of only two values \( x_1 \) and \( x_2 \). However, for the sake of generality, weighted averages of two or more values will be considered in the following development.

To proceed, let (i) \( N \) be an integer \( \geq 2 \), (ii) \( s_\ell \), \( \ell = 1, 2, \ldots, N \), be given positive numbers, and (iii)

\[
w_\ell^\text{def} = \frac{s_\ell}{S} \quad \ell = 1, 2, \ldots, N
\]

where

\[
S^\text{def} = \left( \sum_{\ell=1}^N s_\ell \right) > 0
\]

(Note: to streamline the following development, here we assume that \( s_\ell > 0 \), \( \ell = 1, 2, \ldots, N \), instead of \( s_\ell \geq 0 \), \( \ell = 1, 2, \ldots, N \), as could be inferred from Eq. (4.11). However, without causing any practical harm, one can add a very small positive number, such as \( 10^{-20} \), to each member of a set of nonnegative numbers and turn all of them into positive numbers). It follows from Eqs. (4.12) and (4.13) that

\[
\sum_{\ell=1}^N w_\ell = 1, \quad \text{and} \quad 1 > w_\ell > 0, \quad \ell = 1, 2, \ldots, N
\]

As such,

\[
W^\text{def} = \sum_{\ell=1}^N w_\ell x_\ell
\]

is an “interpolated” weighted average of the real numbers \( x_\ell \). Note that, unless specified otherwise, hereafter \( \ell = 1, 2, \ldots, N \) is assumed.

Let

\[
\delta_\ell^\text{def} = \frac{w_\ell - 1}{N}
\]

Then

\[
w_\ell = \frac{1}{N} + \delta_\ell
\]

Also, with the aid of Eq. (4.14), Eq. (4.16) implies that

\[
\sum_{\ell=1}^N \delta_\ell = 0
\]
Note that \( W \) becomes the simple average of \( x_\ell \) if all \( \delta_\ell = 0 \). Thus the set \( \{ \delta_1, \delta_2, \ldots, \delta_N \} \) provides a measure of how far the weighted average is deviated from the simple average. In the following, a simple way to adjust this deviation will be introduced.

Let

\[
\delta_{\text{min}} \overset{\text{def}}{=} \min \{ \delta_\ell \} \text{ and } \delta_{\text{max}} \overset{\text{def}}{=} \max \{ \delta_\ell \} \tag{4.19}
\]

Then Eq. (4.17) and the fact that \( 1 > w_\ell > 0 \) for all \( \ell \) imply that

\[
1 > \frac{1}{N} + \delta_{\text{max}} \text{ and } \frac{1}{N} + \delta_{\text{min}} > 0 \tag{4.20}
\]

Let some \( \delta_\ell \neq 0 \) (i.e., the case with all \( w_\ell = 1/N \) is excluded). Then Eq. (4.18) implies that \( \delta_{\text{max}} > 0 > \delta_{\text{min}} \). The last inequality and Eq. (4.20) can be combined to yield

\[
1 - \frac{1}{N} > \delta_{\text{max}} > 0 > \delta_{\text{min}} > -\frac{1}{N} \quad \text{(some } \delta_\ell \neq 0) \tag{4.21}
\]

Note that an immediate result of Eq. (4.21) is

\[
\sigma_{\text{max}} \overset{\text{def}}{=} \min \left\{ \frac{1}{\delta_{\text{max}}} (1 - \frac{1}{N}), \frac{1}{N\delta_{\text{min}}} \right\} > 1 \quad \text{(some } \delta_\ell \neq 0) \tag{4.22}
\]

Given any adjustable real parameter \( \sigma > 0 \), let

\[
\delta'_\ell \overset{\text{def}}{=} \sigma \delta_\ell \tag{4.23}
\]

Then Eq. (4.18) implies that

\[
\sum_{\ell=1}^{N} \delta'_\ell = 0 \tag{4.24}
\]

In turn Eq. (4.24) and

\[
w'_\ell \overset{\text{def}}{=} \frac{1}{N} + \delta'_\ell \tag{4.25}
\]

imply that

\[
\sum_{\ell=1}^{N} w'_\ell = 1 \tag{4.26}
\]

As such, \( w'_\ell, \ell = 1, 2, \ldots, N \), form a new set of weight factors. From Eqs. (4.23) and (4.25) one also concludes that the disparity of the weight factors (i.e., the deviation of the values of the weight factors from \( 1/N \)) will be amplified (reduced) if \( \sigma > 1 \) (\( \sigma < 1 \)).

The condition that

\[
1 \geq w'_\ell \geq 0 \tag{4.27}
\]

will be imposed in the current development. With the aid of Eqs. (4.23) and (4.25), and the original assumption that \( \sigma > 0 \), it can be shown that, for the case that some \( \delta_\ell \neq 0 \), Eq. (4.27) is satisfied if and only if

\[
\sigma_{\text{max}} \geq \sigma > 0 \quad \text{(some } \delta_\ell \neq 0) \tag{4.28}
\]

where \( \sigma_{\text{max}} \) is defined in Eq. (4.22). On the other hand, for the special case in which all \( \delta_\ell = 0 \), one can conclude that \( w'_\ell = 1/N \) (and thus Eq. (4.27) is satisfied) for all \( \ell \) and all \( \sigma > 0 \).

Note that, according to Eq. (4.22), \( \sigma_{\text{max}} > 1 \). Moreover, \( \sigma_{\text{max}} \) increases as \( |\delta_{\text{max}}| \) and \( |\delta_{\text{min}}| \) decrease. In fact, \( \sigma_{\text{max}} \to +\infty \) as \( \delta_{\text{max}} \to 0^+ \) and \( \delta_{\text{min}} \to 0^- \). Thus the range of the values of \( \sigma \) allowed becomes larger when \( |\delta_{\text{max}}| \) and \( |\delta_{\text{min}}| \) become smaller. Note that, when \( W \) defined in Eq. (4.15) almost becomes a simple average (i.e., when \( |\delta_{\text{max}}| \ll 1 \) and \( |\delta_{\text{min}}| \ll 1 \), the disparity of the weight factors must be amplified sharply such that the weighted average

\[
W' \overset{\text{def}}{=} \sum_{\ell=1}^{N} w'_\ell x_\ell \tag{4.29}
\]

will deviate substantially from the simple average. In this case, the large range of the values of \( \sigma \) allowed meets the need to use a large value of \( \sigma \). In practice, the value of \( \sigma \) used is that generated using a preset formula as long as the generated value is less than or equal to \( \sigma_{\text{max}} \). For the case that the generated value is larger than \( \sigma_{\text{max}} \), \( \sigma = \sigma_{\text{max}} \) is assumed.

To be more specific, consider the \( N = 2 \) case in which \( x_\ell \) and \( s_\ell, \ell = 1, 2 \), are defined by Eqs. (4.9) and (4.11). It was explained earlier that, for this case, \( w_1 \to 1/2 \) and \( w_2 \to 1/2 \) as \( \nu \to 0 \). In other words, the weighted average \( w_1 x_1 + w_2 x_2 \) almost becomes a simple average when \( |\nu| \ll 1 \). To prevent this from happening, the weight factors \( w_1 \) and \( w_2 \), respectively, are replaced by the new weight factors \( w'_1 \) and \( w'_2 \) generated assuming

\[
\sigma = \min \left\{ \sigma_{\text{max}}, \frac{\sigma_o}{|\nu|} \right\} \tag{4.30}
\]

where \( \sigma_o > 0 \) is a preset parameter in the order of 1. Note that Eq. (4.30) states that (i) \( \sigma = \sigma_o/|\nu| \) if \( \sigma_{\text{max}} \geq \sigma_o/|\nu| \), and (ii) \( \sigma = \sigma_{\text{max}} \) if \( \sigma_{\text{max}} < \sigma_o/|\nu| \). As such, \( \sigma \gg 1 \) when \( |\nu| \ll 1 \).

Note that, for any \( N = 2 \) case, one of \( \delta_1 \) and \( \delta_2 \) is \( \delta_{\text{max}} \) while another is \( \delta_{\text{min}} \). As a result, Eqs. (4.18), (4.20), and (4.22) imply that

\[
0 < \delta_{\text{max}} = -\delta_{\text{min}} < 1/2 \quad \text{(some } \delta_\ell \neq 0) \tag{4.31}
\]

and

\[
\sigma_{\text{max}} = \frac{1}{2 \delta_{\text{max}}} \quad \text{(some } \delta_\ell \neq 0) \tag{4.32}
\]
Also for any case with \( N = 2 \), \( \delta_{\text{max}} > 0 \) and \( \sigma = \sigma_{\text{max}} \), it can be shown that: (i) \( w_1 = 1 \) and \( w_2 = 0 \) if \( \delta_1 = \delta_{\text{max}} \), and (ii) \( w_1 = 1 \) and \( w_2 = 0 \) if \( \delta_2 = \delta_{\text{max}} \).

This completes the description of a new approach by which the weight factors \( w_\ell \), \( \ell = 1, 2, \ldots, N \), are generated from the given weight factors \( w_\ell \), \( \ell = 1, 2, \ldots, N \). In the following, another approach will be described.

To proceed, the indices of \( s_\ell \), \( \ell = 1, 2, \ldots, N \), will be reshuffled such that

\[
s_N \geq s_{N-1} \geq \ldots \geq s_1 > 0 \quad (4.33)
\]

As such, Eqs. (4.12) and (4.13) imply that Eq. (4.14) can be replaced by a set of stronger conditions, i.e.,

\[
\sum_{\ell=1}^{N} w_\ell = 1 \quad \text{and} \quad 1 > w_N \geq w_{N-1} \geq \ldots \geq w_1 > 0
\]  

Next let

\[
\eta_\ell \overset{\text{def}}{=} \frac{s_{\ell+1}}{s_\ell} - 1, \quad \ell = 1, \ldots, N-1
\]  

Then (i) \( \eta_\ell \geq 0 \), \( \ell = 1, \ldots, N-1 \), and (ii)

\[
s_{\ell+1} = \left[ \prod_{\ell'=1}^{\ell} (1 + \eta_{\ell'}) \right] s_1, \quad \ell = 1, \ldots, N-1
\]  

Given any adjustable real parameter \( \sigma > 0 \), let (i) \( \tilde{s}_1 = s_1 \) and

\[
\tilde{s}_{\ell+1} = \left[ \prod_{\ell'=1}^{\ell} (1 + \sigma \eta_{\ell'}) \right] \tilde{s}_1, \quad \ell = 1, \ldots, N-1
\]  

and (ii)

\[
\tilde{w}_\ell \overset{\text{def}}{=} \frac{\tilde{s}_\ell}{\tilde{S}}, \quad \ell = 1, 2, \ldots, N
\]  

where

\[
\tilde{S} \overset{\text{def}}{=} \left( \sum_{\ell=1}^{N} \tilde{s}_\ell \right) > 0
\]  

Because \( \sigma > 0 \) and \( \eta_\ell \geq 0 \), \( \ell = 1, \ldots, N-1 \), Eq. (4.37) implies that

\[
s_N \geq \tilde{s}_{N-1} \geq \ldots \geq \tilde{s}_1 > 0
\]  

Also, as a result of Eqs. (4.38)–(4.40), one has

\[
\sum_{\ell=1}^{N} w_\ell = 1 \quad \text{and} \quad 1 > w_N \geq w_{N-1} \geq \ldots \geq w_1 > 0
\]  

As such, \( \tilde{w}_\ell \), \( \ell = 1, 2, \ldots, N \), form a new set of weight factors and

\[
\tilde{W} \overset{\text{def}}{=} \sum_{\ell=1}^{N} \tilde{w}_\ell x_\ell
\]  

is an "interpolated" weighted average of the real numbers \( x_\ell \). Note that, for the special case that \( s_N = s_{N-1} = \ldots = s_1 > 0 \), it is easy to see that (i) \( w_N = \tilde{w}_N = 1/N \), \( \ell = 1, 2, \ldots, N \), and (ii) \( \eta_\ell = 0 \), \( \ell = 1, \ldots, N-1 \).

Let \( \ell_1 \) and \( \ell_2 \) be any pair of integers with \( 1 \leq \ell_1 < \ell_2 \leq N \). Then Eqs. (4.12) and (4.36)–(4.38) imply that

\[
\frac{w_{\ell_2}}{w_{\ell_1}} = \prod_{\ell' = 1}^{\ell_1 - 1} (1 + \eta_{\ell'})
\]

and

\[
\frac{\tilde{w}_{\ell_2}}{\tilde{w}_{\ell_1}} = \prod_{\ell' = 1}^{\ell_2 - 1} (1 + \sigma \eta_{\ell'})
\]

Because \( \sigma > 0 \) and \( \eta_\ell \geq 0 \), \( \ell = 1, \ldots, N-1 \), a comparison of Eqs. (4.43) and (4.44) reveals that \( w_{\ell_2}/w_{\ell_1} = \tilde{w}_{\ell_2}/\tilde{w}_{\ell_1} = 1 \) if \( \eta_\ell = 0 \) for all \( \ell \) with \( \ell_1 \leq \ell \leq (\ell_2 - 1) \). However, in case that \( \eta_\ell \neq 0 \) for at least one \( \ell \) with \( \ell_1 \leq \ell \leq (\ell_2 - 1) \), one has

\[
\frac{\tilde{w}_{\ell_2}}{\tilde{w}_{\ell_1}} = \begin{cases} 
> \frac{w_{\ell_2}}{w_{\ell_1}} & \text{if } \sigma > 1 \\
< \frac{w_{\ell_2}}{w_{\ell_1}} & \text{if } \sigma < 1
\end{cases}
\]  

From the above discussions, one concludes that, except for the special case in which \( s_N = s_{N-1} = \ldots = s_1 \), the disparity of \( \tilde{w}_\ell \) is greater (less) than that of \( w_\ell \) if \( \sigma > 1 \) (\( \sigma < 1 \)). Note that the current approach for amplifying the weight factors has one advantage over the approach described earlier, i.e., in the current approach, there is no upper bound for the value of \( \sigma \) one could use. Thus, in the current approach, Eq. (4.30) is simplified as

\[
\sigma = \frac{\sigma_o}{|\nu|}
\]  

4.4. Schemes w-3 and w-4

Consider the \( N = 2 \) case in which \( x_\ell \) and \( s_\ell \), \( \ell = 1, 2 \), are defined by Eqs. (4.9) and (4.11). Let \((w_\ell)^n \) and \((w_\ell)^f \), respectively, be the weight factors associated with \((\tilde{u}_\ell^-)^n \) and \((\tilde{u}_\ell^+)^n \) generated using the first approach described in Sec. 4.3. Then, by definition, Scheme w-3 is formed by Eq. (2.8) and

\[
(u_\ell)^n = (w_\ell)^n (\tilde{u}_\ell^-)^n + (w_\ell^f)^n (\tilde{u}_\ell^+)^n
\]  

On the other hand, let \((\tilde{w}_\ell^-)^n \) and \((\tilde{w}_\ell^+)^n \), respectively, be the weight factors associated with \((\tilde{u}_\ell^-)^n \) and
and \((\hat{u}_{x+})_j^n\) generated using the second approach described in Sec. 4.3. Then, by definition, Scheme w-4 is formed by Eq. (2.8) and
\[
(\hat{u}_x)_j^n = (\hat{w}_-)_j^n (\hat{u}_{x-})_j^n + (\hat{w}_+)_j^n (\hat{u}_{x+})_j^n \quad (4.48)
\]

5. Schemes for nonuniform meshes

Consider the space-time mesh \((-L \leq x \leq L)\) with \(t \geq 0\) depicted in Fig. 5. Here (i) \(L > 0\) is a given length, and (ii) the mesh structure in the region \(-L \leq x < 0\) is the mirror image of that in the region \(0 < x \leq L\).

Let the domain \(0 \leq x \leq L\) on the \(x\)-coordinate line be divided into \(K\) intervals using the dividing coordinate points \(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{K-1}\) where
\[
0 < \hat{x}_1 < \hat{x}_2, \ldots, \hat{x}_{K-1} = L \quad (5.1)
\]

Let
\[
L_k \overset{\text{def}}{=} \hat{x}_k - \hat{x}_{k-1}, \quad k = 1, 2, \ldots, K \quad (5.2)
\]
with
\[
\hat{x}_0 \overset{\text{def}}{=} 0 \quad \text{and} \quad \hat{x}_K \overset{\text{def}}{=} L \quad (5.3)
\]

Then
\[
\sum_{k=1}^{K} L_k = L, \quad \text{and} \quad L_k > 0, \quad k = 1, 2, \ldots, K \quad (5.4)
\]

In the current development, it is assumed that
\[
L_k = r^{k-1} L_1, \quad k = 1, 2, \ldots, K \quad (5.5)
\]

where \(r\) is a parameter with
\[
r > 0 \quad \text{and} \quad r \neq 1 \quad (5.6)
\]

By combining Eqs. (5.4)–(5.6), one has
\[
L_k = \frac{(1-r)^{k-1} L}{1-r^K}, \quad k = 1, 2, \ldots, K \quad (5.7)
\]

In turn, Eqs. (5.2), (5.3), and (5.7) imply that
\[
\hat{x}_k = \left(\frac{1-r^k}{1-r^K}\right) L, \quad k = 0, 1, 2, \ldots, K \quad (5.8)
\]

Note that, unless specified otherwise, hereafter it is assumed that \(k = 1, 2, \ldots, K\).

Moreover, for any \(k\), let the interval \((\hat{x}_{k-1}, \hat{x}_k)\) be divided into \(M\) uniform sub-intervals with the dividing points \(\hat{x}_{k-1}^{(1)}, \hat{x}_{k-1}^{(2)}, \ldots, \hat{x}_{k-1}^{(M-1)}\) where \(M > 0\) is a given integer and
\[
\hat{x}_{k-1}^{(1)} < \hat{x}_{k-1}^{(2)} < \cdots < \hat{x}_{k-1}^{(M-1)} < \hat{x}_k \quad (5.9)
\]

Thus
\[
\hat{x}_{k-1}^{(m)} - \hat{x}_{k-1}^{(m-1)} = \ell_k, \quad m = 1, 2, \ldots, M \quad (5.10)
\]

where
\[
\ell_k \overset{\text{def}}{=} \frac{L_k}{M} = \frac{(1-r)^{k-1}}{1-r^K} \left\{ \frac{L}{M} \right\} \quad (5.11)
\]

and
\[
\hat{x}_{k-1}^{(0)} \overset{\text{def}}{=} \hat{x}_{k-1} \quad \text{and} \quad \hat{x}_{k-1}^{(M)} \overset{\text{def}}{=} \hat{x}_k \quad (5.12)
\]

Note that (i) the validity of the last equality sign in Eq. (5.11) follows from Eq. (5.7); and (ii) Eqs. (5.8)–(5.12) imply that
\[
\hat{x}_k^{(m)} = \frac{L}{1-r^K} \left\{ 1 + \left[ 1 + (r-1) \frac{m}{M} \right] r^k \right\}, \quad k = 0, 1, \ldots, K-1; \quad m = 0, 1, \ldots, M \quad (5.13)
\]

Note that \(\hat{x}_k^{(0)}\) is yet to be defined. However, as a natural extension of Eq. (5.12) (i.e., let \(k = K+1\) be allowed in the first part of Eq. (5.12)), it is assumed that \(\hat{x}_K^{(0)} = \hat{x}_K\). According to Eq. (5.8), \(\hat{x}_K = L\). Thus
\[
\hat{x}_K^{(0)} \overset{\text{def}}{=} L \quad (5.14)
\]

As will be shown, the above definition is needed in a later development. Note that, unless specified otherwise, hereafter it is assumed that \(m = 1, 2, \ldots, M\).

To pave the way, let
\[
R(f) \overset{\text{def}}{=} \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{KM}{2} \right\} \quad (5.15)
\]

\[
D(f) \overset{\text{def}}{=} \{(K, 0)\} \cup \{(k, m)|k = 0, 1, \ldots, K-1; \quad m = 0, 1, \ldots, M-1\} \quad (5.16)
\]

By definition, (i) \(R(f)\) represents a set of integers and half-integers and (ii) \(D(f)\) represents a set of ordered pairs of integers. It can be shown easily that there are \((KM+1)\) elements in each of these two sets. In addition, the function \(f\) defined by
\[
f = f(k, m) \overset{\text{def}}{=} \frac{km + M}{2} \quad (5.17)
\]

represents a one-to-one mapping between \(D(f)\) and \(R(f)\). For the special case \(K = 3\) and \(M = 4\), the values of the function \(f\) are listed in Table I. For any given \(j \in R(f)\), let (i) \((k, m) = f^{-1}(j)\) where \(f^{-1}\) is the inverse of \(f\); and (ii)
\[
x_j \overset{\text{def}}{=} \hat{x}_k^{(m)} \quad \text{and} \quad x_{-j} \overset{\text{def}}{=} -x_j \quad (5.18)
\]

Hereafter, for any \(j = 0, \pm 1/2, \pm 1, \ldots, \pm KM/2,\) the mesh line \(x = x_j\) will be referred to as the \(j\)th spatial mesh line.
Next, the intersection of the $j$th spatial mesh line and the horizon line with $t = n\Delta t$ ($n = 0, 1/2, 1, \ldots$) is considered as a mesh point (marked by a dot in Fig. 5) and denoted by $(j, n)$ if $(j + n)$ is a half-integer. By its definition, the coordinates of $(j, n)$ are given by

$$x = x_j \quad \text{and} \quad t = t^n \overset{\text{def}}{=} n\Delta t \quad (5.19)$$

In this section, the set of all $(j, n)$ is again denoted by $\Omega$. As shown in Fig. 6(a,b), (i) each $(j, n) \in \Omega$ is associated with two conservation element $\text{CE}_-(j, n)$ and $\text{CE}_+(j, n)$; and (ii)

$$\Delta x_j^- \overset{\text{def}}{=} x_j - x_{j-1/2} \quad \text{and} \quad \Delta x_j^+ \overset{\text{def}}{=} x_{j+1/2} - x_j \quad (5.20)$$

As a result, $\Delta x_j^- = \Delta x_{j-1/2}^-$ and $\Delta x_j^+ = \Delta x_{j+1/2}^+$. Also, as shown in Fig. 6(c),

$$\text{CE}(j, n) \overset{\text{def}}{=} \text{CE}_-(j, n) \cup \text{CE}_+(j, n) \quad (5.21)$$

and

$$\Delta x_j \overset{\text{def}}{=} x_{j+1/2} - x_{j-1/2} = \Delta x_j^- + \Delta x_j^+ \quad (5.22)$$

Hereafter, the midpoint of the top face of $\text{CE}(j, n)$ is referred to as the solution point (marked by a cross in Fig. 6(c)) associated with the mesh point $(j, n)$ and is denoted by $(j, n)'$. As shown in Fig. 5, a solution point may or may not coincide with a mesh point. Also, as depicted in Fig. 6(d), each $(j, n)'$ is associated with a solution element $\text{SE}(j, n)$.

Note that, by its definition, the coordinates of $(j, n)'$ are given by

$$x = x_j' \overset{\text{def}}{=} \frac{1}{2}(x_j + x_{j+1/2}) \quad \text{and} \quad t = t^n \quad (5.23)$$

In addition, the midpoint of the line segment joining $(j, n)$ and $(j + 1/2, n)$, and that of line segment joining $(j, n)$ and $(j - 1/2, n)$ are marked by circles (see Fig. 6(c)) and, respectively, denoted by $(j, n)'^+$ and $(j, n)'^-$. Note that the line segment joining $(j, n)$ and $(j + 1/2, n)$, and that joining $(j, n)$ and $(j - 1/2, n)$ are the top faces of $\text{CE}_+(j, n)$ and $\text{CE}_-(j, n)$, respectively. As such, $(j, n)'^+$ and $(j, n)'^-$ are the current counterparts of points $M_+$ and $M_-$ depicted in Fig. 3. By their definitions, the coordinates of $(j, n)'^+$ and $(j, n)'^-$ are given by

$$x = x_j'^+ \overset{\text{def}}{=} \frac{1}{2}(x_j + x_{j+1/2}) \quad \text{and} \quad t = t^n \quad (5.24)$$

and

$$x = x_j'^- \overset{\text{def}}{=} \frac{1}{2}(x_j + x_{j-1/2}) \quad \text{and} \quad t = t^n \quad (5.25)$$

respectively.

Furthermore, by using Eqs. (5.20), (5.23) and (5.24), it can be shown that

$$x_j - x_{j-1/2} = \frac{1}{2}\Delta x_j^- > 0 \quad (5.26)$$

$$x_j^{+1/2} - x_j = \frac{1}{2}\Delta x_j^+ > 0 \quad (5.27)$$

$$x_j^{+1/2} - x_j = \frac{1}{2}\Delta x_j^+ > 0 \quad (5.28)$$

$$x_j - x_{j-1/2} = \frac{1}{2}\Delta x_j^- > 0 \quad (5.29)$$

$$x_j^{+1/2} - x_j = \frac{1}{2}\Delta x_j^+ > 0 \quad (5.30)$$

$$x_j^{+1/2} - x_j = \frac{1}{2}\Delta x_j^+ > 0 \quad (5.31)$$

$$x_j^{+1/2} - x_j = \frac{1}{2}\Delta x_j^+ > 0 \quad (5.32)$$

The relations Eqs. (5.20), and (5.22)–(5.32) are clearly illustrated in Fig. 7.

The current counterparts of points $P_+$ and $P_-$ depicted in Fig. 3 are points $P_+$ and $P_-$ depicted in Fig. 7. By definition, the coordinates of points $P_+$ and $P_-$, respectively, are given by

$$x = x(P_+) \quad \text{and} \quad t = t^n \quad (5.33)$$

and

$$x = x(P-) \quad \text{and} \quad t = t^n \quad (5.34)$$

with

$$x(P_+) \overset{\text{def}}{=} x_j^{+} + \tau_j (x_{j+1/2}^{+} - x_j^{+}) = x_j^{+} + \frac{\tau_j}{2}\Delta x_j^{+1/2}$$

$$= x_j^{+1/2} - \frac{1 - \tau_j}{2}\Delta x_j^{+1/2} \quad (5.35)$$

$$x(P-) \overset{\text{def}}{=} x_j^{+} - \tau_j \left(x_{j-1/2}^{+} - x_j^{+}\right) = x_j^{+} - \frac{\tau_j}{2}\Delta x_j^{-1/2}$$

$$= x_j^{+1/2} - \frac{1 - \tau_j}{2}\Delta x_j^{-1/2} \quad (5.36)$$

Here note that: (i) $\tau_j$, with the range

$$0 \leq \tau_j \leq 1 \quad (5.37)$$

is a parameter to be defined later; and (ii) as the value of $\tau_j$ increases from 0 to 1, $P_+$ will move away from $(j, n)^+$ and edge toward the solution point $(j + 1/2, n)'$ while $P_-$ will move away from $(j, n)^-$ and edge toward the solution point $(j - 1/2, n)'$.

This completes the specification of key geometric parameters. Next note that, the current counterpart to Eq. (2.3) is

$$u^n(x, t; j, n) \overset{\text{def}}{=} u^n_j + (u_x^n)_j (x - x_j') + (u_t^n)_j (t - t^n) \quad (5.38)$$
i.e., \( u^n_j \) is the value of \( u^* \) at the solution point \((j, n)\). Because \( (u^n_j) = -a(u^n_j) \) is also assumed here, the current counterpart to Eq. (2.5) is

\[
u^n(x, t; j, n) = u^n_j + (u^n_j) \left[(x-x^n_j) - a(t-t^n)\right]
\]

(5.39)

Obviously, Eq. (2.4) is valid even for the current case. By imposing Eq. (2.11), and using Eqs. (2.4) and (5.39), one can obtain the current counterpart to Eq. (2.8), i.e.,

\[
u^n_j = \frac{\Delta x^+}{\Delta x_j} \left[u^n_{j+1/2} + (u^n_j) x^n_j + x^n_{j+1/2} \right] \\
+ \frac{\Delta x^n}{\Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
- \frac{\alpha \Delta t}{2 \Delta x_j} \left[u^n_{j+1/2} + (u^n_j) x^n_j + x^n_{j+1/2} \right] \\
- \frac{\alpha \Delta t}{4 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
+ \frac{\alpha \Delta t}{2 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
- \frac{\alpha \Delta t}{4 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
(5.40)
\]

Note that, normalized numerical analogues of \( \partial u/\partial x \) are not used in this section.

Next, let

\[
u^*(P_{+j}) \equiv u^* (x(P_{+j}), t^n; j + 1/2, n - 1/2)
\]

(5.41)

and

\[
u^*(P_{-j}) \equiv u^* (x(P_{-j}), t^n; j - 1/2, n - 1/2)
\]

(5.42)

By using Eqs. (5.35), (5.36), and (5.39), Eqs. (5.41) and (5.42) imply

\[
u^*(P_{+j}) = u^n_{j+1/2} + \\
[(1 - \tau_j)x^n_{j+1/2} - x^n_{j+1/2}(-a\Delta t/2)](u^n_{j+1/2}) \\
(5.43)
\]

and

\[
u^*(P_{-j}) = u^n_{j-1/2} + \\
[(1 - \tau_j)x^n_{j-1/2} - x^n_{j-1/2}(-a\Delta t/2)](u^n_{j-1/2}) \\
(5.44)
\]

respectively. Eqs. (5.43) and (5.44) are the current counterparts to Eqs. (3.6) and (3.7), respectively.

Next, the current counterparts to Eqs. (4.1) and (4.2) are

\[
u^n_j = \frac{\Delta x^+}{\Delta x_j} \left[u^n_{j+1/2} + (u^n_j) x^n_j + x^n_{j+1/2} \right] \\
+ \frac{\Delta x^n}{\Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
- \frac{\alpha \Delta t}{2 \Delta x_j} \left[u^n_{j+1/2} + (u^n_j) x^n_j + x^n_{j+1/2} \right] \\
- \frac{\alpha \Delta t}{4 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
+ \frac{\alpha \Delta t}{2 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
- \frac{\alpha \Delta t}{4 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
(5.45)
\]

and

\[
u^n_j = \frac{\Delta x^+}{\Delta x_j} \left[u^n_{j+1/2} + (u^n_j) x^n_j + x^n_{j+1/2} \right] \\
+ \frac{\Delta x^n}{\Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
- \frac{\alpha \Delta t}{2 \Delta x_j} \left[u^n_{j+1/2} + (u^n_j) x^n_j + x^n_{j+1/2} \right] \\
- \frac{\alpha \Delta t}{4 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
+ \frac{\alpha \Delta t}{2 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
- \frac{\alpha \Delta t}{4 \Delta x_j} \left[u^n_{j-1/2} + (u^n_j) x^n_j - x^n_{j-1/2} \right] \\
(5.45)
\]

respectively. Note that it can be shown easily that the denominators in Eqs. (5.45) and (5.46) are positive always. By using one of many approaches described earlier, one can generate \((u^n_j)^n\) and \((u^n_j)^n\), the weight factors associated with \((\bar{u}_{x^n})_j^n\) and \((u^n_j)_j^n\). Given \((u^n_j)^n\) and \((u^n_j)^n\), a solver can be formed by Eq. (5.40) and

\[
u^n_j = (u^n_j)^n (\bar{u}_{x^n})_j^n + (u^n_j)^n (\bar{u}_{x^n})_j^n \\
(5.47)
\]

To complete the description of the current solvers, note that, among many possible choices, one may assume that

\[
\tau_j = \frac{|a\Delta t/e_j|, \quad j = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots, \pm (KM - 1)/2 \quad (5.48)
\]

where \(e_j\) are positive constants defined by: (i)

\[
e_j \equiv \min\{(x^n_j - x^n_{j-1}), (x^n_{j+1} - x^n_{j})\} \quad (5.49a)
\]

if \(j = 0, \pm 1/2, \pm 1, \ldots, \pm (KM - 2)/2\); (ii)

\[
e_j \equiv \min\{2(x^n_j - x^n_{j-1}), (x^n_{j+1} - x^n_{j})\} \quad (5.49b)
\]

if \(j = -(KM - 1)/2\); and (iii)

\[
e_j \equiv \min\{2(x^n_{j+1} - x^n_{j}), (x^n_{j} - x^n_{j-1})\} \quad (5.49c)
\]

if \(j = (KM - 1)/2\)

Note that \(\tau_j\) given in Eq. (5.48) can be interpreted as the \(CFL\) number at \((j, n)\). In other words, Eq. (5.48) is the current counterpart of a special case of Eq. (3.15) in which \(\delta = s\). Also Note that, because \(x_{(KM+1)/2}\) and \(x_{-(KM+1)/2}\) are undefined, \(x_{\pm KM/2}\) cannot defined using Eq. (5.23). However, according to Eq. (5.49), \(x_{\pm KM/2}\) are needed in the definitions of \(e_{\pm (KM-1)/2}\). As such, here we assume that

\[
x^n_{\pm KM/2} = x_{\pm KM/2} \\
(5.50)
\]

Also, to satisfy Eq. (5.37) for all values of \(j\), \(\Delta t\) is subjected to the condition that

\[
\Delta t \leq \frac{\min\{e_j\}}{|a|} \quad (5.51)
\]

We conclude this section with a discussion of Eq. (5.11) for the special case in which \(r^K \gg 1\). For the case \(r^K \gg 1\), Eq. (5.11) implies that

\[
\ell_K = \frac{(1-r)r^{K-1}}{1-r^K} \frac{L}{M} \approx \frac{(1-r)r^{K-1}}{-r^K} \frac{L}{M}
\]
\[ r = \frac{r - 1}{r} \cdot \frac{L}{M} \quad (r^K \gg 1) \quad (5.52) \]

In turn, Eq. (5.52) implies that
\[ r \approx \frac{1}{1 - M(\ell_K/L)} \quad (r^K \gg 1) \quad (5.53) \]

Finally, by combining Eqs. (5.7) and (5.53), one has
\[ R \equiv \frac{L_K}{L_I} = r^{K-1} \approx \left( \frac{1}{1 - M(\ell_K/L)} \right)^{K-1} \quad (r^K \gg 1) \quad (5.54) \]

Thus, for the case \( r^K \gg 1 \), \( r \) and therefore \( R \) can be easily evaluated in terms of \( M \) and the ratio \( \ell_K/L \) without using an iterative procedure, as would be expected if Eq. (5.11) is used directly.

6. Two-dimensional extensions using triangular meshes

Several 2D CE/SE schemes using triangular meshes were described in [7.8]. These 2D schemes also can be easily modified to become \( CFL \) number insensitive schemes. A sketch explaining how this can be done is given here.

Consider Fig. 8. Here, a triangle \( \triangle ABC \) is surrounded by three neighboring triangles \( \triangle ADB \), \( \triangle BEC \) and \( \triangle CFA \). The centroids of \( \triangle ABC \), \( \triangle ADB \), \( \triangle BEC \), and \( \triangle CFA \) are denoted by \( G \), \( G_1 \), \( G_2 \), and \( G_3 \), respectively.

Let the plane shown in Fig. 8 be an \( x-y \) plane at the time level \( t = t^n \), \( n = 0, 1/2, 1, \ldots \). Then \( G \) can be considered as a mesh point at \( t = t^n \). As shown in [7.8], the three conservation elements associated with \( G \) are three quadrilateral space-time cylinders sandwiched between the time level at \( t = t^n \) and that at \( t = t^{n-1/2} \) with their top faces being \( GAG_1B \), \( GBG_2C \), and \( GCCG_3A \), respectively. The centroids of the above three top faces are denoted by \( M_1 \), \( M_2 \) and \( M_3 \), respectively. \( M_1 \), \( M_2 \), and \( M_3 \) are the current counterparts to points \( (j, n)^+ \) and \( (j, n)^- \) depicted in Fig. 7 (or points \( M_4 \) and \( M_5 \) depicted in Fig. 3).

Furthermore, let the solution point associated with point \( G \) be the centroid of the hexagon \( AG_1BG_2CG_3 \) (which is the union of the three top faces referred to in the last paragraph) and denoted by \( G' \) (not shown in Fig. 8). In general, \( G' \) does not coincide with \( G \). Also \( G \) and \( G' \) are current counterparts to \( (j, n) \) and \( (j, n)' \) depicted in Figs. 6 and 7. The solution points associated with \( G_1 \), \( G_2 \), and \( G_3 \) can be defined similarly and denoted by \( G'_1 \), \( G'_2 \), and \( G'_3 \), respectively. The projections of \( G'_1 \), \( G'_2 \), and \( G'_3 \) at the time level \( t = t^{n-1/2} \) are the current counterparts to \( (j + 1/2, n - 1/2)' \) and \( (j - 1/2, n - 1/2)' \) depicted in Fig. 7.

In addition, note that points \( P_1 \), \( P_2 \), and \( P_3 \) depicted in Fig. 8 are the current counterparts to points \( P_{ij} \) and \( P_{-j} \) depicted in Fig. 7. By definition, they lie on the line segments \( M_1G_1, M_2G_2, \) and \( M_3G_3 \), respectively. (Note: in Fig. 8, \( G'_1, G'_2, \) and \( G'_3 \) happen to coincide with \( G_1, G_2 \) and \( G_3 \), respectively.) It is also assumed that
\[ \frac{M_1P_i}{M_1G_i} = \tau(G_i), \quad i = 1, 2, 3 \quad (6.1) \]

where \( \tau(G_i) \) is the current counterpart to \( \tau_j \) (see Eq. (5.48)) and is subjected to the condition
\[ 0 \leq \tau(G) \leq 1 \quad (6.2) \]

Finally note that, points \( M_1 \), \( M_2 \), and \( M_3 \) were incorrectly defined in [36] as the midpoints of the line segments \( G'/G'_1, G'/G'_2, \) and \( G'/G'_3 \), respectively.

7. Numerical results

The solvers described in this paper, in a straightforward manner described in [3,7,8], have been extended to become their Euler versions. Accuracy of these Euler solvers will be assessed here.

7.1 Sod’s Shock-tube problem

First, accuracy of the 1D uniform-mesh Euler versions of the special scheme described in Sec. 2.3 and Scheme w-4 are evaluated using Sod’s shock tube problem [41]. The results are shown in Figs. 9 and 10 with the understanding that the “old” solutions are generated using the special scheme with \( \alpha = 1 \) (see Eq. (2.39)) while the new solutions are generated using Scheme w-4 with \( \alpha = 1 \) and \( \sigma_0 = 0.5 \) (see Eqs. (4.11) and (4.46)). Here the spatial domain is defined by \(-0.505 \leq x \leq 0.505 \) with \( \Delta x = 0.01 \). The numerical solutions at \( t = 0.2 \), including density \( (\rho) \), velocity \( (u) \) and pressure \( (p) \), are presented for two different sizes of time step, i.e., (i) \( \Delta t = 4 \times 10^{-3} \) (Fig. 9); and (ii) \( \Delta t = 4 \times 10^{-6} \) (Fig. 10). The values of the global \( CFL \) number for these two cases approximately are 0.88 and 0.00088, respectively. Because each marching step advances a time period \( \Delta t/2 \), it requires 100 and 100,000 marching steps for cases (i) and (ii), respectively, to advance to \( t = 0.2 \).

From the results shown, it is clear that the old CE/SE solutions are highly dissipative when the value of global \( CFL \) number \( \ll 1 \). In sharp contrast, the new solutions are still quite accurate even when \( CFL \) number \( < 0.001 \). The advantage of Scheme w-4 over the scheme described in Sec. 2.3 is overwhelming.

Next, accuracy of the 1D nonuniform-mesh Euler versions of Scheme w-3 (\( \alpha = 2 \) and \( \sigma_0 = 1 \)) are also evaluated using Sod’s shock tube problem. The results obtained using three different nonuniform meshes are shown in Figs. 11-13. Using the
geometric parameters defined in Sec. 5, the three meshes, respectively, are defined by (i) $L = 0.5$, $r = 1/0.96$, $M = 4$, and $K = 170$; (ii) $L = 0.5$, $r = 2$, $M = 50$, and $K = 11$; and (iii) $L = 0.5$, $r = 4$, $M = 15$, and $K = 6$. By using Eqs. (5.53) and (5.54), it is easy to shown that (i) $t_k / L = 0.01$ for all three cases, and (ii) $R \approx 1000$ for case (i) and $R = 1024$ for cases (ii) and (iii). The numerical results ($t = 0.2$) are generated after 100,000 marching steps assuming $\Delta t = 4 \times 10^{-6}$.

Because $R = t_k / t_1 \approx 1000$, the variation in mesh intervals is very large for all three cases. As a result, the disparity in CFL numbers is also very large—they vary from 0.88 near $x = 0$ to 0.00088 near $x = \pm L$. Nevertheless, according to Figs. 11-13, the numerical results match very well with the exact solutions for all three cases. The only exception is that, for cases (ii) and (iii) where $r = 2$ and $r = 4$, respectively, there are small solution bumps near interfaces separating mesh intervals of vastly different sizes. It is interesting to note that solution accuracy at points away from these interfaces are not impacted by these bumps.

7.2 Propagation of Sound Waves through a Transonic Nozzle

The benchmark problem 1.1 in the third Computational Aeroacoustics (CAA) Workshop that have been solved using the 2D CE/SE Euler solver with adjustable numerical dissipation in [44] are used to test the 2D Euler version of scheme w-2 described above. The same 401 stretching mesh used in [44] is used here. As shown in Fig. 14, the current numerical results agree very well with the exact solutions.

7.3 Oblique shock reflection

The oblique shock reflection problem[7] is solved using 9600 (120x80x2) uniformly-distributed structured triangles with $\Delta t = 0.005$ and 0.0005, respectively. For $\Delta t = 0.005$, the Courant number is in the order of 1. Numerical results obtained using different models are very similar. However, for $\Delta t = 0.0005$, the solution obtained using an old CE/SE scheme becomes very dissipative. In contrast, as shown in Fig. 15, the numerical solutions obtained using the 2D Euler version of scheme w-2 still preserve sharp shock resolution.

8. Conclusions and discussions

Generally speaking, a stable numerical marching for a non-linear problem requires the presence of a sufficient amount of numerical dissipation. However, accuracy of the numerical results, especially for an unsteady problem, will suffer if too much numerical dissipation is present. As such, a careful control of numerical dissipation is a must for an accurate and stable non-linear unsteady numerical simulation. However, a proper control of numerical dissipation is a very difficult task. Although one can increase the numerical dissipation rather easily, it is much harder to reduce it when accuracy consideration requires it.

The CE/SE method is developed from a set of non-dissipative solvers. As such each CE/SE solver is an extension of a core non-dissipative scheme. It is this unique feature that make it much easier to reduce numerical dissipation in a CE/SE simulation. It is also the key reason behind the successful construction of the Courant number insensitive CE/SE schemes described in this paper.

In this paper, the ideas used in a recent work on one-dimensional CFL number insensitive schemes are elaborated in a much more detailed manner. These ideas are also extended to construct one-dimensional and multidimensional solvers for applications involving nonuniform meshes. The advantages of the new schemes over the original CE/SE schemes are overwhelming and clearly demonstrated by the numerical results presented.

In addition, we also report advances in developing new wiggle-suppressing techniques. These new techniques are based on a new set of weighted averaging formulae which are much more potent and flexible than those introduced earlier.

Finally note that, in addition to the Euler applications reported in this paper and [35,36], the CFL number insensitive schemes recently have also been applied to (i) Navier-Stokes problems by C.L. Chang [42]; and (ii) chromatographic adsorption problems [43] by Y.I. Lim et al..

References


3. S.C. Chang, The method of space-time conser-


37. Other CE/SE references are posted on: http://www.grc.nasa.gov/www/microbus.


42. Chau-Lyan Chang, private communication.


Table I. The values of function $f(K=3, M=4)$

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Figure 1. — A surface element on the boundary $S(V)$ of an arbitrary space-time volume $V$. 

20
Figure 2. — The SEs and CEs.
Figure 3.—Definition of points $P_{-1}$, $M_{m}$, $M_{+}$, and $P_{+}$.

Figure 4. The function $\tau_0(s)$.

Figure 5.—A space-time mesh with nonuniform spatial intervals ($K = M = 2$).
Figure 6.—The SEs and CEs for a nonuniform mesh. (a) CE \( (j, n) \). (b) CE \( (j, n)^+ \). (c) CE \( (j, n) \). (d) SE \( (j, n) \).

Figure 7.—Key geometric parameters and their relations.
Figure 8.—Definition of points $P_1, P_2, P_3, M_1, M_2,$ and $M_3$.

Figure 9: Uniform-mesh solutions to Sod’s problem (CFL number = 0.88).
Figure 10: Uniform-mesh solutions to Sod’s problem (CFL number = 0.00088).
Figure 11: Nonuniform-mesh solutions to Sod’s problem ($L = 0.5$, $r = 1/0.96$, $M = 4$, $K = 170$).
Figure 12: Nonuniform-mesh solutions to Sod’s problem ($L = 0.5, r = 2, M = 50, K = 11$).
Figure 13: Nonuniform-mesh solutions to Sod's problem \((L = 0.5, r = 4, M = 75, K = 6)\).
Figure 14: CE/SE solution of the acoustic wave using a 401 point nonuniform mesh.

Figure 15: CE/SE solution of the oblique shock problem on a uniform mesh with $\Delta t = 0.0005$. 