The Effect of a Tangential Contact Force Upon the Rolling Motion of an Elastic Sphere on a Plane

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The motion and deformation of an elastic sphere rolling on an elastic plane under a normal contact pressure N have been studied for the case where a tangential force T is also sustained at the point of contact. Provided that $T < \mu N$ (μ = coefficient of friction), the sphere rolls without sliding but exhibits a small velocity relative to the plane, termed "creep." Following the work of Mindlin and Poritsky, it is shown that creep arises from slip over part of the area of contact, and further, that this slip takes place toward the trailing edge of the contact area. On the assumption of a locked region in which no slip occurs, of circular shape, tangential to the circle of contact at its leading point, surface tractions are found which satisfy the condition of no slip within the locked region and are approximately consistent with the laws of friction in the slip region. The variation of creep velocity with tangential force is thereby determined. Experimental measurements of the creep of a steel ball rolling on a flat steel surface are in reasonable agreement with the theoretical results.

Introduction

This paper and its sequel[2] examine the motion of an elastic sphere which rolls without sliding on an elastic plane, taking into account the deformation of the two surfaces under the action of the contact force exerted between them.

A pressure between the surfaces, acting normal to the plane, has the effect of enlarging the "point of contact" into a small circular area. In addition, the surfaces may sustain a tangential component of force without sliding, provided that the limiting friction force is not exceeded. Thus as rolling proceeds, the line of action of the resultant force transmitted between the sphere and plane may be inclined to the normal at an angle not exceeding the angle of friction for the two surfaces.

To define the kinematic aspects of the problem, it has been found convenient to regard the motion relative to the point of contact, which remains stationary in space. The origin O of rectangular co-ordinate axes is taken to be at the center of the circle of contact. Axes Ox and Oy lie in the undistorted plane, with Ox in the direction of rolling; Oz is therefore normal to the plane, Fig. 1. In this view a steady rolling motion of the sphere on the plane appears as a stationary pattern of elastic distortion around the contact region through which the material of both bodies flows at a steady rate. This will be recognized to be the Eulerian co-ordinate system, familiar in the field of hydrodynamics, but which has been applied to problems in elasticity by Bishop and Goodier (1).[3]

Focusing attention on a particular point lying within the contact area, slip is said to be taking place at that point if the velocities with which the two surfaces pass through that point are unequal. If there is no point in the contact area at which the velocities of the two surfaces are identical, then complete slip or sliding[4] is taking place.

The particle velocities at any point are made up of two components; the first due to the rigid-body motions of the sphere or plane, and the second arising from the pattern of strain through which the material is flowing. The rigid-body motions corresponding to steady rolling are shown in Fig. 1. The plane has a linear velocity U (the rolling velocity) parallel to Ox. The sphere has an angular velocity ω about a diametral axis parallel to Oy, and in addition may have an angular velocity Ω relative to the plane about the normal axis Oz. This latter motion is referred to as "spin," and is usually associated with rolling along a curved path.

From the previous discussion there emerge three distinct categories into which the rolling problem may be classified:

Free rolling, in which the resultant force transmitted between the contact surfaces is perpendicular to the plane, and further

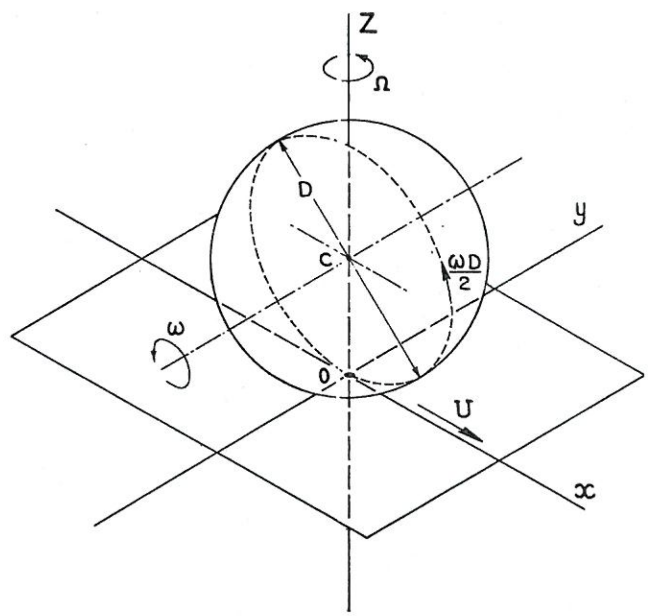

Fig. 1 Rolling of a sphere on a plane; the co-ordinate system

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[2] "The Effect of Spin Upon the Rolling Motion of an Elastic Sphere on a Plane," by K. L. Johnson, see preceding paper in this issue of the Journal of Applied Mechanics, pp. 332–338.

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[3] Numbers in parentheses refer to the Bibliography at the end of the paper.

[4] The term "sliding" is used to denote the condition of complete bodily slip, while "slip" refers to the condition at a specified point, or points, in the contact area.

there is no angular velocity of spin between the sphere and the plane.

Rolling under tangential forces, which takes place when the transmitted force is not normal to the plane, but contains a tangential component. The tangential component may be resolved into the direction of rolling (longitudinal) and at right angles to that direction (transverse).

Rolling with spin, which occurs when the sphere and the plane move with an angular velocity relative to each other, about an axis normal to the plane.

The small resistance to motion in free rolling has been attributed by some, notably Osborne Reynolds (2), Heathcote (3), and Eldredge (4), to slip over the area of contact arising from the geometry of the indentation of the surface on which the ball rolls. In developing this theory, tangential displacements of the contact surfaces produced by frictional tractions are ignored, even though their magnitude is of the same order as the displacements due to normal pressure which define the shape of the indentation.

An alternative explanation of the energy dissipated in free rolling lies in the influence of the (so-called) elastic hysteresis of the material of the two surfaces. Palmgren (5) favors this view and, more recently, Tabor (6) has shown that careful rolling-friction measurements are consistent with the elastic hysteresis hypothesis.

This paper is not concerned with the resistance in free rolling except for the fact it is unavoidably present in experiments which are described.

It has long been appreciated that the effect of a tangential component of force between two bodies rolling together gives rise to a slow velocity of one surface relative to the other in a direction opposed to the tangential force acting upon it. The phenomenon, usually referred to as creep, is well understood in relation to the motion of a belt over a pulley, clearly described by Swift (7). A significant feature, common to all examples of creep during rolling, is the fact that the area of contact is divided into two distinct regions; one in which the surfaces move together without relative velocity and the other over which slip occurs. The creep velocity arises from the difference in state of strain between the two surfaces over that part of the area of contact where they remain locked together.

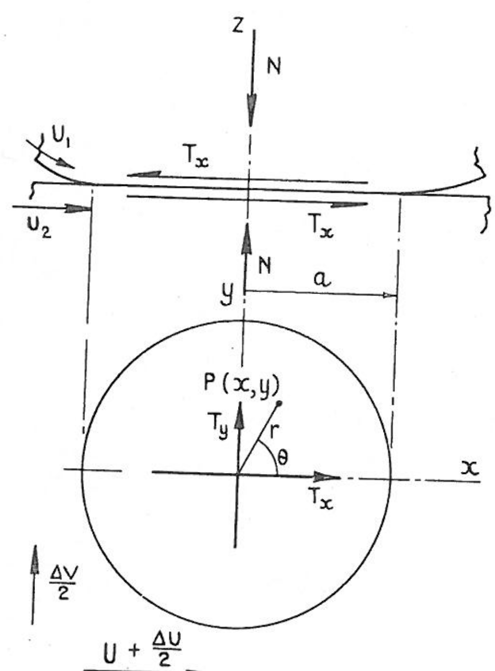

Fig. 2 Area of contact showing tangential-force components T_x and T_y. Creep velocities are denoted by ΔU and ΔV.

Identical solutions have been found by Carter (8) and Poritsky (9) for the two-dimensional problem of two cylinders rolling together with a normal force N' and a tangential force T' per unit length acting between them. Owing to the normal pressure the cylinders are in contact over a strip of width $2b$ given by the Hertz theory. If μ is the coefficient of friction between the surfaces, Poritsky shows that the cylinders remain locked together with no slip over a strip of width

$$2d = 2b\,(1 - T'/\mu N')^{1/2} \qquad [1]$$

from which it follows that when $T' = 0$ no slip occurs, and when $T' = \mu N'$ slip has penetrated over the entire contact surface. In the discussion Cain (10) demonstrates that the locked region must have one boundary coincident with the leading edge of the area of contact, and that slip occurs after contacting points have rolled through the locked region, if the direction of slip is to be consistent with the laws of friction and to oppose the tangential force acting between the surfaces.

An approximate numerical solution for the rolling of a ball on a curved surface under tangential force has been obtained by Palmgren (11) which involves approximating the elliptical area of contact to a rectangle. For the case evaluated, the analysis compares favorably with experimental results, but it is an unfortunate feature of his experiments that spin is present in addition to tangential forces.

Statement of the Problem

The ball is pressed onto the plane by a normal force N resulting in a circular area of contact of radius a over which, by the Hertz theory, the pressure is distributed according to

$$Z = \frac{3N}{2\pi a^2}\left(1 - \frac{r^2}{a^2}\right)^{1/2} \qquad [2]$$

where

$$a^3 = \frac{3(1-\nu)ND}{8G} \qquad [3]$$

for a ball of diameter D rolling on a flat surface of similar elasticity, where G = modulus of rigidity and ν = Poisson's ratio.

The area of contact is shown in Fig. 2. The path of rolling is parallel to the x-axis so that material is flowing through the area of contact in the positive direction with a mean rolling velocity U. A tangential force T, having components T_x and T_y, is transmitted across the surface ($T < \mu N$). As a result of this force the sphere has component velocities of creep relative to the plane denoted by ΔU and ΔV. It is required to find the distribution of tangential tractions X and Y over the surface of contact resulting from these tangential forces, and the corresponding values of the creep velocities ΔU and ΔV.

As in the Hertz theory, it is proposed to neglect the slight warping of the contact surface, so that it may be taken to be a plane area having a circular boundary of radius given by Equation [3]. Further, if the sphere and plane are assumed to have the same elastic properties, the problem is equivalent to that of two identical spheres rolling together with their circular area of contact lying centrally in the x-y plane.

The theoretical problem for which a solution is sought concerns, therefore, a system of two bodies rolling together which are geometrically and elastically symmetrical with respect to the three co-ordinate axes. Several important conditions follow from this symmetry.[5] A further simplification is secured by considering separately the cases where T_x and T_y act alone.

[5] The effect of the warping of the contact surface and unequal elastic constants upon the results so obtained is discussed subsequently.

Conditions Arising From Symmetry

With T_x acting alone, reference to Fig. 2 shows that if the lower sphere is rotated through 180 deg about Oy it becomes identical with the upper sphere, provided the direction of U is reversed. If (x_1, y_1) and (x_2, y_2) are two points in contact, after rotation about Oy, $x_2 = -x_1$ and $y_2 = y_1$. Equal and opposite tractions $X_2 = -X_1$ and $Y_2 = -Y_1$ are transmitted between the two surfaces at this point. The tractions themselves may be expressed

$$X = f(x, y, \operatorname{sgn} U) \text{ and } Y = f(x, y, \operatorname{sgn} U) \qquad [4]$$

Remembering that rotation about Oy causes a change in sign of X but not in Y, the symmetry of the two bodies about Oy gives

$$X(-x, y, -U) = X(x, y, U) \qquad [5]$$

and

$$Y(-x, y, -U) = -Y(x, y, U) \qquad [6]$$

The over-all magnitude of the tractions X and Y is governed by the conditions of equilibrium

$$T_x = \int\int X\, dx\, dy, \quad T_y = \int\int Y\, dx\, dy \qquad [7]$$

These conditions apply over the whole area of contact whether slip occurs or not.

Conditions which must be satisfied by the tangential surface displacements if no slip is to occur also follow from symmetry. The components of surface velocity at the point $P(x, y)$, denoted by q_x and q_y, may be expressed

$$q_x = f(x, y, U) \text{ and } q_y = f(x, y, U)$$

If (x_1, y_1) and (x_2, y_2) are two points instantaneously in contact, the components of relative velocity of slip between them are denoted by

$$s_x = q_{x1} - q_{x2} \quad \text{and} \quad s_y = q_{y1} - q_{y2}$$

With T_x acting alone, symmetry about Oy gives

$$q_{x1}(-x, y, -U) = -q_{x2}(x, y, U)$$

and

$$q_{y1}(-x, y, -U) = q_{y2}(x, y, U)$$

whereupon

$$s_x = q_x(x, y, U) + q_x(-x, y, -U) \qquad [8]$$

and

$$s_y = q_y(x, y, U) - q_y(-x, y, -U) \qquad [9]$$

It is now necessary to express the velocities q_x and q_y in terms of the rigid-body motions, and the local velocity components arising from the pattern of strain through which the material is flowing. In the Eulerian co-ordinate system the particle velocities are given by (1)

$$q_x = U + \left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)u$$

$$q_y = V + \left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)v$$

where U and V are the rigid-body velocities, and where u and v are the tangential elastic displacement at P.

In view of the symmetry, the creep velocities ΔU and ΔV may be divided equally between the two bodies, so that

$$U_1 = U + \frac{\Delta U}{2}, \qquad U_2 = U - \frac{\Delta U}{2}$$

$$V_1 = \frac{\Delta V}{2}, \qquad V_2 = -\frac{\Delta V}{2}$$

Now ΔU and ΔV are small compared with U, furthermore, for steady rolling u and v do not vary with time, so that the expressions for q_x and q_y reduce to

$$q_x = U + \frac{\Delta U}{2} + U\frac{\partial u}{\partial x}(x, y, \operatorname{sgn} U) \qquad [10]$$

$$q_y = \frac{\Delta V}{2} + U\frac{\partial v}{\partial x}(x, y, \operatorname{sgn} U) \qquad [11]$$

Expressions [8] and [9] for the relative velocity components become

$$s_x = \Delta U + U\frac{\partial u}{\partial x}(x, y, U) - U\frac{\partial u}{\partial x}(-x, y, -U) \qquad [12]$$

and

$$s_y = \Delta V + U\frac{\partial v}{\partial x}(x, y, U) + U\frac{\partial v}{\partial x}(-x, y, -U) \qquad [13]$$

Over any region in which there is no slip taking place the relative velocities s_x and s_y must of necessity be zero whence

$$\frac{\partial u}{\partial x}(-x, y, -U) - \frac{\partial u}{\partial x}(x, y, U) = \frac{\Delta U}{U} \equiv \xi_x = \text{const.} \qquad [14]$$

and

$$-\frac{\partial v}{\partial x}(-x, y, -U) - \frac{\partial v}{\partial x}(x, y, U) = \frac{\Delta V}{U} \equiv \xi_y = \text{const.} \qquad [15]$$

Equations [14] and [15], therefore, provide conditions which must be satisfied by the strain components $\partial u/\partial x$ and $\partial v/\partial x$ over that part of the surface of contact in which no slip takes place, and at the same time enable the magnitude of the creep-velocity ratios ξ_x and ξ_y to be calculated.

It is to be expected, however, that over part, at least, of the contact area slip will in fact take place. In this region Equations [14] and [15] will not be satisfied but alternative conditions may be obtained by an appeal to the simple laws of friction. It is assumed that the resultant tangential traction Q can nowhere exceed the normal traction Z multiplied by a constant coefficient of friction, and that in the region where slip has occurred Q is taken to have its limiting value μZ; i.e., within the locked region

$$|Q| \equiv (X^2 + Y^2)^{1/2} < \frac{3\mu N}{2\pi a^2}\left(1 - \frac{r^2}{a^2}\right)^{1/2} \qquad [16]$$

while on the boundary of the locked region and within the slip region

$$|Q| = \frac{3\mu N}{2\pi a^2}\left(1 - \frac{r^2}{a^2}\right)^{1/2} \qquad [17]$$

The direction of Q at any point in the slip region is determined in fact by the direction of the relative velocity between the surfaces at that point, which it must oppose. Since this direction is not known in advance, the expedient will be used of choosing the direction of Q a priori to coincide with the direction of the tangential force; i.e.,

$$\left.\begin{array}{ll}\text{when } T_x \text{ acts alone} & Y = 0 \\ \text{and when } T_y \text{ acts alone} & X = 0\end{array}\right\} \qquad [18]$$

The direction of relative sliding at any point associated with this assumed traction may be calculated subsequently by the use of Equations [12] and [13] in order to confirm that the chosen direction of Q does not contradict the physical law of friction that it should oppose the direction of relative slip.

A further difficulty is that the shape of the boundary which divides the locked region from the slip region is not known in advance. In consequence, it has been found necessary to follow a "trial-and-error" approach. A shape for the locked region is assumed at the outset which, together with specified tractions over the area of contact, enables the tangential surface displacements to be found. For the assumed solution to be correct, the displacements must satisfy the no-slip conditions [14] and [15] within the locked region and the laws of friction in the slip region.

To complete the boundary conditions it is necessary to specify the normal traction (or displacement) in the contact area. Physically, it is required that the bodies should remain in contact and in equilibrium across the contact surface, for which w and Z' must be continuous across that surface. Z' denotes the normal traction resulting from the tangential force only, and is given by

$$Z' = -\frac{2G}{1-2\nu}\left[(1-\nu)\left(\frac{\partial w}{\partial z}\right)_{z=0} + \nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_{z=0}\right] \ldots\ldots [19]$$

In obtaining a solution to the problem of static contact between spheres Mindlin (13) makes the assumption that $Z' = 0$, while showing that this is only exactly true for no slip (whence $\partial u/\partial x = \partial v/\partial y = 0$) and for bodies of equal elastic constants (where continuity of w and $\partial w/\partial z$ across the contact surface gives $\partial w/\partial z = 0$). The assumption that $Z' = 0$ will be maintained here, whether slip occurs or not.

In consequence of the symmetry one sphere only need be considered which, following Hertz, is taken to be a semi-infinite solid. The problem in elasticity therefore reduces to the "problem of the plane" in which the surface tractions are given over the whole boundary. To summarize the boundary conditions:

1 In the locked region, X and Y to be specified, consistent with Equations [5], [6], and [16]; $Z' = 0$.

2 In the slip region, X and Y given by Equations [17] and [18]; $Z' = 0$.

3 Outside the area of contact ($r > a$); $X = Y = Z' = 0$.

The solution to the boundary-value problem in elasticity defined in the foregoing is obtained by the methods employed by Cattaneo (12) and Mindlin (13) in studying the effects of a tangential force acting between two spheres in static (as opposed to rolling) contact. Reference should also be made to Love (14) (page 242).

Static Contact

Mindlin shows first that the tractions

$$X = \frac{T_x}{2\pi a}\,\frac{1}{(a^2 - r^2)^{1/2}}, \quad Y = 0 \ldots\ldots\ldots\ldots [20]$$

produce displacements[6] over the circle of contact

$$u = \text{constant} = \frac{(2-\nu)T_x}{4Ga}, \quad v = 0 \ldots\ldots\ldots [21]$$

which correspond to no slip over the whole of the contact area, under the action of a static force T_x.

Examining this solution for the case of rolling contact we see that the tractions of Equations [20] satisfy the symmetry conditions [5] and [6], and since $\partial u/\partial x = \partial v/\partial x = 0$, the displacements satisfy the no-slip conditions [14] and [15] with the result $\xi_x = \xi_y = 0$; i.e., no creep occurs. It would appear therefore that the tractions [20] and the displacements [21] also provide a solution to the rolling problem, for materials of perfect elasticity and with all slip prevented. It is interesting to note that the exclusion of a region of slip leads to zero creep velocities. This result might be expected from energy considerations, since slip provides the only mechanism in an elastic solid to account for the external work done by the tangential force when creep occurs.

It is obvious that only an academic solution has been found, since the infinite traction at the boundary of the contact area ($r = a$) insures that some slip will in fact take place. If slip occurs in a direction parallel to the x-axis over the whole contact area, then the tractions are taken to be

$$X = \frac{3\mu N}{2\pi a^3}(a^2 - r^2)^{1/2}, \quad Y = 0 \ldots\ldots\ldots\ldots [22]$$

which are shown to give rise to surface displacements

$$\left.\begin{aligned} u &= \frac{3\mu N}{64Ga^3}\left[2(2-\nu)(2a^2 - x^2 - y^2) + \nu(x^2 - y^2)\right] \\ v &= \frac{3\mu N}{64Ga^3}\,2\nu xy \end{aligned}\right\} \ldots [23]$$

Cattaneo's device of adding tractions

$$X' = -\left(\frac{a'}{a}\right)^3 \frac{3\mu N}{2\pi a'^3}(a'^2 - r^2)^{1/2}, \quad Y' = 0 \ldots\ldots [24]$$

over the circle $0 < r < a'$ results in total displacements

$$\left.\begin{aligned} u + u' &= \frac{3\mu N(2-\nu)}{16Ga}\left(1 - \frac{a'^2}{a^2}\right) = \text{const} \\ v + v' &= 0 \end{aligned}\right\} \; 0 < r < a' . [25]$$

Thus the combination of tractions [22] and [24] results in no slip over the circle $0 < r < a'$. This result defines the locked region as a circle concentric with the circle of contact and the slip region as an annulus whose width increases with tangential force. Experimental evidence of an annular region of slip in static contact has been obtained by Mindlin, Mason, et al. (15), and Johnson (16).

Rolling Contact

(a) *Longitudinal Force.* At first sight it would appear that the combined tractions [22] and [24], together with the corresponding displacements, provide a solution to the case of rolling contact. Both tractions are even in x, satisfying condition [5], and nowhere exceed μZ, satisfying [16].

In the locked region $\partial u/\partial x = \partial v/\partial x = 0$ which satisfy conditions [14] and [15] with zero creep. It will be shown, however, that the remaining condition is not satisfied; the direction of relative motion does not oppose the frictional traction throughout the annulus of slip.

The displacements outside the contact area ($r > a$) due to the traction of Equation [22] acting alone have been evaluated previously (17) with the result[7]

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &\simeq -\frac{3\mu N(2-\nu)}{16Ga^2}\,\frac{x}{a}\,\frac{2}{\pi}\left[\sin^{-1}\frac{a}{r} - \frac{a}{r}\left(1 - \frac{a^2}{r^2}\right)^{1/2}\right] \\ \frac{\partial v}{\partial x} &\simeq 0 \end{aligned}\right\} \ldots [26]$$

[6] In what follows, u and v, without subscript, denote displacements in the plane $z = 0$.

[7] The approximation involves neglecting terms whose order of magnitude is $\nu/2(2-\nu)$, i.e., 0.09, compared with unity.

Within the contact area ($r < a$), from Equations [23][8]

$$\frac{\partial u}{\partial x} = -\frac{3\mu N}{32Ga^3}(4 - 3\nu)x, \quad \frac{\partial v}{\partial x} = -\frac{3\mu N}{32Ga^3}\nu y \ldots [27]$$

Replacing a by a' in Equations [26] and [27] and multiplying by the factor $-(a'/a)^3$ gives the values of $\partial u'/x$ and $\partial v'/\partial x$, resulting from the traction X'. These expressions, together with the resultant value of $\partial u/\partial x$ due to $X + X'$, are sketched, for $y = 0$, in Fig. 3(a). It will be seen that in the annulus of slip $\partial u/\partial x$ is $+ve$ for $x < 0$, and $-ve$ for $x > 0$. Over the locked region $\partial u/\partial x = \partial v/\partial x = 0$, from which it follows (see Equations [14] and [15]) that the creep velocities ΔU and ΔV are zero.

The relative velocity components between the two surfaces at any point are given by Equations [12] and [13]. Noting from Equation [26] that $\partial u/\partial x$ is an odd function of x, and that $\partial v/\partial x = 0$, it follows that

$$s_x = 2U\frac{\partial u}{\partial x}(x, y, U), \quad s_y = 0 \ldots\ldots\ldots\ldots [28]$$

The traction X is positive over the whole of the annulus of slip, while the relative velocity of slip is also positive over the region $x < 0$, a result which entirely contradicts the physical law of friction.

In order to obtain a more correct solution to the problem of rolling contact it is necessary to redefine the locked region. Following the two-dimensional solution of Poritsky, the example of creeping belts and also the foregoing results, we are led to expect slip to take place after the contacting points have passed through the locked region. As a tentative solution, let it be assumed that the locked region remains a circle of radius a' but is moved in a position so as to be tangential to the contact circle at the leading point $(-a, 0)$, Fig. 3(b). Its center would then be at the point $(-c, 0)$ where

$$c = a - a' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [29]$$

The traction X and the accompanying displacements of Equations [23] remain as before, but the added traction becomes

$$X' = -\left(\frac{a'}{a}\right)^3 \frac{3\mu N}{2\pi a'^3}(a'^2 - r'^2)^{1/2} \ldots\ldots\ldots [30]$$

where

$$r'^2 = (x + c)^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots [31]$$

This traction results in

$$\frac{\partial u'}{\partial x} = \frac{3\mu N}{32Ga^3}(4 - 3\nu)(x + c), \quad \frac{\partial v'}{\partial x} = \frac{3\mu N}{32Ga^3}\nu y \ldots [32]$$

which, when added to Equations [27], give over-all values

$$\frac{\partial u}{\partial x} = \frac{3\mu N(4 - 3\nu)c}{32Ga^3}, \quad \frac{\partial v}{\partial x} = 0 \ldots\ldots\ldots\ldots [33]$$

Noting that for negative values of U the locked region should be tangential to the circle of contact at $(a, 0)$, so that c becomes negative, Equations [33] satisfy the no-slip conditions [14] and [15] in the assumed locked region, with the result

$$\xi_x = \frac{\Delta U}{U} = -\frac{3\mu N(4 - 3\nu)c}{16Ga^3}, \quad \xi_y = 0 \ldots\ldots [34]$$

The creep ratio in the x-direction is in this case no longer zero. The value of a' can be obtained from the equilibrium condition [7], which gives

$$\frac{T_x}{\mu N} = 1 - \left(\frac{a'}{a}\right)^3 \ldots\ldots\ldots\ldots\ldots\ldots [35]$$

[8] Making the same approximation as in Equation [26] would give

$$\frac{\partial u}{\partial x} \simeq -\frac{3\mu N(2 - \nu)x}{16Ga^3} \quad \text{and} \quad \frac{\partial v}{\partial x} \simeq 0$$

which are consistent with Equation [26] at $r = a$.

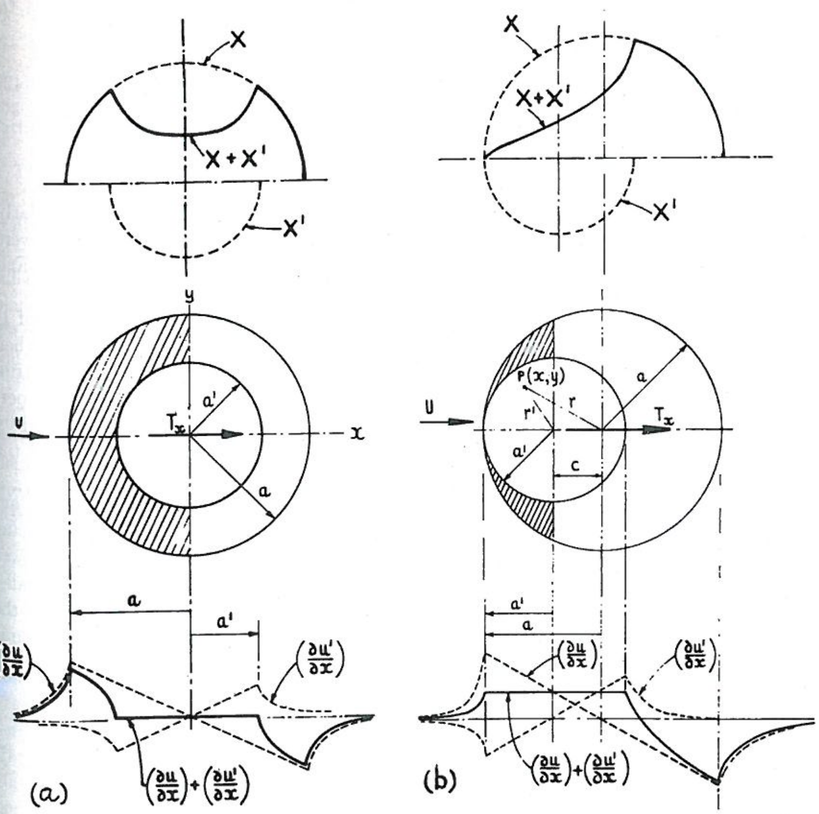

Fig. 3 **(a) Mindlin's solution for static contact under tangential force. Slip takes place in the annulus $a' < r < a$. Relative slip velocity is proportional to $(\partial u/\partial x + \partial u'/\partial x)$ which is positive in shaded area. (b) Modified solution for rolling contact. Locked region is taken to be circle radius a'. $(\partial u/\partial x + \partial u'/\partial x)$ is constant, corresponding to no slip, over this circle. Slip velocity remains positive in shaded area.**

whence

$$\frac{c}{a} = 1 - \frac{a'}{a} = 1 - \left(1 - \frac{T_x}{\mu N}\right)^{1/3} \quad \ldots\ldots\ldots [36]$$

Substituting in [34] gives an expression for the creep ratio in terms of the tangential force; viz.

$$\xi_x = -\frac{3\mu N(4 - 3\nu)}{16Ga^2}\left[1 - \left(1 - \frac{T_x}{\mu N}\right)^{1/3}\right] \quad \ldots\ldots [37]$$

It will be seen that the combined tractions X over the slip region and $X + X'$ over the locked region satisfy the required conditions of Equations [5] and [16].

It remains to examine the direction of relative slip outside the locked region. In this case, because of the traction X'

$$\left.\begin{aligned} \frac{\partial u'}{\partial x} &= \frac{3\mu N(2 - \nu)}{16Ga^2}\,\frac{x + c}{a}\,\frac{2}{\pi} \\ &\qquad \left[\sin^{-1}\frac{a'}{r'} - \frac{a'}{r'}\left(1 - \frac{a'^2}{r'^2}\right)^{1/2}\right] \\ \frac{\partial v'}{\partial x} &= 0 \end{aligned}\right\} \quad \ldots [38]$$

in the region $r' > a'$. Thus the net value of $\partial u/\partial x$ in the region of slip is given by adding Equations [27] and [38], and is drawn for $y = 0$ in Fig. 3(*b*). The relative velocity between the two surfaces in this region is found from Equations [12] and [13], which give

$$\left.\begin{aligned} s_x &= -\frac{3\mu N(2 - \nu)}{8Ga^2}\,\frac{x + c}{a} \\ &\qquad \left[1 - \frac{2}{\pi}\sin^{-1}\frac{a'}{r'} + \frac{2}{\pi}\,\frac{a'}{r'}\left(1 - \frac{a'^2}{r'^2}\right)^{1/2}\right] \\ s_y &= 0 \end{aligned}\right\} \quad \ldots [39]$$

The value of s_x given by this expression is found to be positive and hence in the same direction as the traction in the shaded area of Fig. 3(*b*). The existence of an area in which the law of friction is contravened implies that the assumed circular shape of the locked region cannot be correct. The shaded "area of error" is, however, in no event large compared with the total area of applied traction (< 8 per cent) and becomes small both when $T_x \rightarrow 0$ and when $T_x \rightarrow \mu N$, so that the foregoing solution should represent a reasonable approximation.

(*b*) *Transverse Force.* The transverse creep ξ_y associated with a transverse tangential force T_y may be obtained without difficulty by the same method. Mindlin's solution for static contact gives a concentric circular locked region, with tangential tractions which are everywhere positive and parallel to the tangential force T_y. Applying this solution to the rolling problem and investigating the direction of slip in the annulus $a' < r < a$, again results in slip velocities which are approximately parallel to the tangential traction (i.e., parallel to oy) and positive in the shaded region of Fig. 3(*a*), where $x < 0$. We were led, therefore, to expect that in the case of a transverse force also, slip will take place at the trailing edge and not at the leading edge of the circle of contact. As before, the locked region is taken to be a circle tangential to the leading edge at the point $(-a, 0)$.

With T_y acting alone, the system is symmetrical about ox, which leads to conditions similar to [5] and [6] which must be satisfied by the surface tractions, and to conditions similar to [14] and [15] which must be satisfied by the surface displacements in the region of no slip.

It may be shown that the tractions

$$\left.\begin{aligned} X + X' &= 0 \\ Y + Y' &= \frac{3\mu N}{2\pi a^2}(a^2 - r^2)^{1/2} \\ &\qquad - \left(\frac{a'}{a}\right)^3 \frac{3\mu N}{2\pi a^3}(a'^2 - r'^2)^{1/2} \end{aligned}\right\} \quad \ldots [40]$$

satisfy the necessary conditions and result in strains

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = \frac{3\mu N(4 - \nu)c}{32Ga^3}$$

over the circle of no slip $r' < a'$. These values correspond to zero longitudinal creep ($\xi_x = 0$) and a transverse creep given by

$$\xi_y = -\frac{3\mu N(4 - \nu)}{16Ga^2}\left[1 - \left(1 - \frac{T_y}{\mu N}\right)^{1/3}\right] \quad \ldots\ldots [41]$$

This result compares with Equation [37] for the longitudinal creep. The distribution of surface tractions is the same in each case and is everywhere parallel to the applied force.

It is still necessary to confirm that the direction of slip opposes the frictional traction. The relative velocity in the slip region has been calculated and, as before, turns out to be approximately parallel to the traction (in this case Y) and of negative sign everywhere except in the shaded portion of Fig. 3(*b*).

The theoretical results just derived have all been obtained for a system comprising identical spheres. The slight warping of the contact surface which occurs when a sphere rolls on a plane or when the two bodies have unequal elastic constants is ignored, but the experience of Tabor (6) suggests that its influence is small. Dissimilar elastic constants make the distribution of equal creep velocities to each surface in Equations [10] and [11] untenable. However, by distributing creep velocities ΔU_1 and $-\Delta U_2$, ΔV_1 and $-\Delta V_2$ to the sphere and plane, respectively, where $\Delta U_1 + \Delta U_2 = \Delta U$ and $\Delta V_1 + \Delta V_2 = \Delta V$, it may be shown that for bodies having elastic constants ν_1, G_1, and ν_2, G_2 the net creep is given by

$$\xi = \frac{1}{2}\left[\xi(\nu_1, G_1) + \xi(\nu_2, G_2)\right] \quad \ldots\ldots\ldots\ldots [42]$$

in which the results of Equations [37] and [41] apply.

Experimental

To check the validity of the theoretical results, particularly in view of the approximations involved, simple experiments were designed to measure the creep due to both longitudinal and transverse tangential forces.

The apparatus is shown in Fig. 4. Two hard steel balls of 1-in. diam were mounted on a spindle by soldered joints. When placed in contact with a plane, in fact, a hard steel parallel strip, the balls were free to roll with equal velocity in a direction perpendicular to the axis of the spindle. The balls were loaded by two hangers mounted on small ball races at each end of the spindle.

A longitudinal tangential component of force at the contact between each ball and the plane was obtained by tilting the plane through an angle β in the direction of rolling. The system was restrained from free rolling down the incline by strings, parallel to the plane, passing over pulleys mounted at the ends of the spindle. Reference to Fig. 4(*a*) shows that the contact force is now inclined at an angle α to the plane, given by

$$\frac{T_x}{N} = \tan\alpha = \frac{R}{R + \frac{1}{2}D}\tan\beta$$

where R is the radius of the pulley.

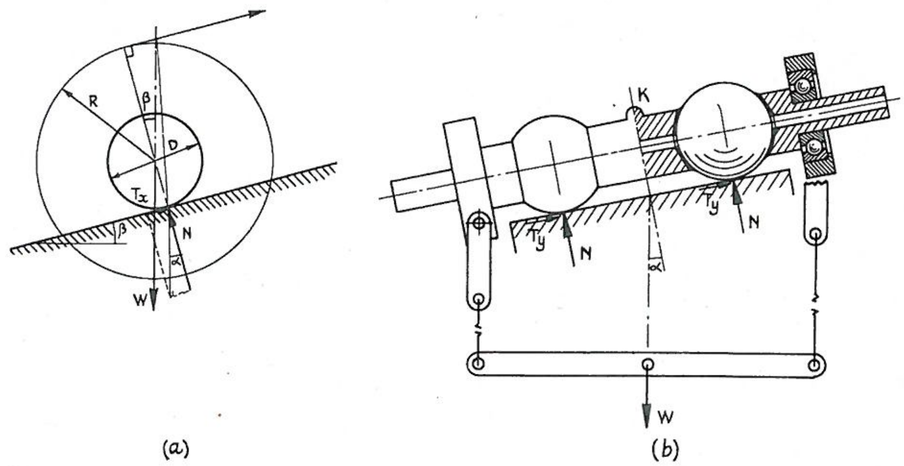

Fig. 4 Experimental apparatus for measuring creep under action of tangential forces. (a) Application of a longitudinal tangential force. (b) Application of a transverse tangential force.

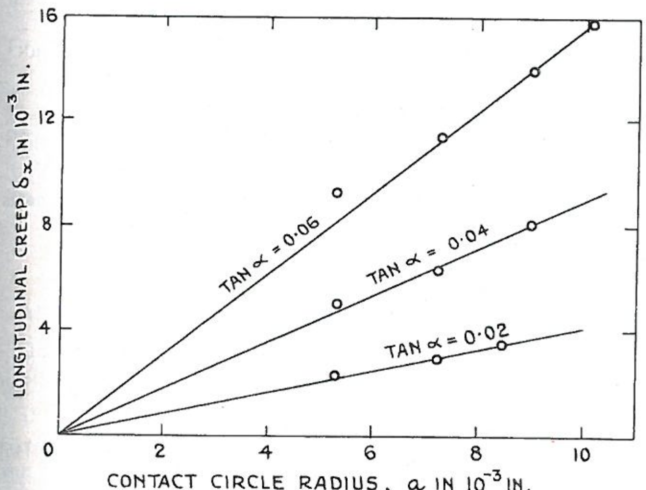

Fig. 5 Experimental variation of creep with increasing area of contact, obtained by increasing normal load

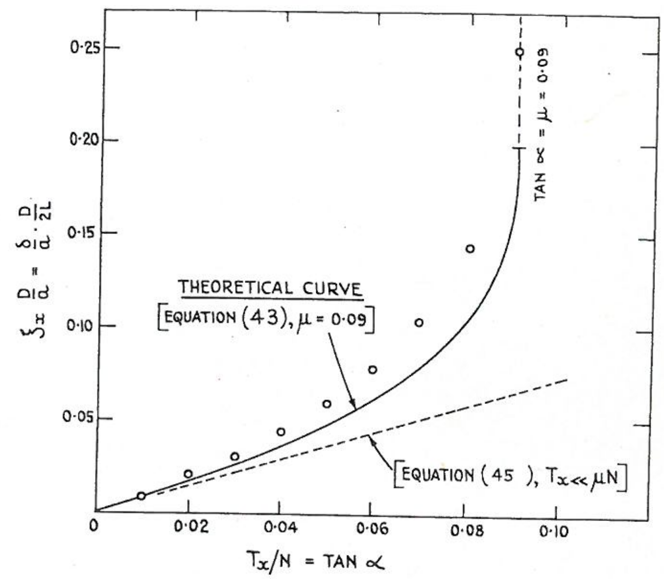

Fig. 6 Comparison of experimental longitudinal creep measurements with theoretical analysis

The system was rolled slowly up the plane for a distance $L = 10$ in. by pulling on the two strings, and then allowed to roll slowly back by gradually releasing the strings. The spindle returned to its initial angular position relative to the plane with the spherical knob K resting in contact with a parallel slip placed on the plane. The longitudinal creep of the system down the plane (denoted by δ) resulting from a traversal up and down the plane was measured by micrometer or, in cases where greater sensitivity was required, by observing a line scratched on the spindle by a microscope mounted above.

Both the balls and the plane possessed a good surface finish. Before each experiment they were cleaned, washed in benzene and ether, and wiped dry.

By use of Equation [3], the Expression [37] for the longitudinal creep may be written

$$\frac{\delta}{2L} = -\xi_x = \frac{\mu(4-3\nu)}{2(1-\nu)}\,\frac{a}{D}\left[1-\left(1-\frac{\tan\alpha}{\mu}\right)^{1/2}\right] \quad ..[43]$$

Thus for a constant value of $\tan\alpha$, the creep should be proportional to a/D. The ball diameter D remained unchanged, but a was varied by increasing the load. The direct proportionality of δ with a is shown by the experimental results plotted in Fig. 5. Macroscopic sliding began on the downward run at $\tan\alpha = 0.09$, compared with values of $\mu = 0.12$–0.14 measured in steady rectilinear sliding. Fig. 6 is plotted from the slopes of the straight lines of Fig. 5 and compared with Equation [43], drawn for $\mu = 0.09$.

The application of a transverse tangential force was obtained by tilting the plane through an angle perpendicular to the direction of rolling as shown in Fig. 4(b). The plane was maintained horizontal in the direction of rolling so that the pulley strings could be dispensed with. The rolling motion was produced by hand so that the velocities of rolling, as in the previous experiment, were very small, certainly less than 10 fpm. The transverse creep displacements across the plane due to a double traversal were measured by micrometer. The procedure was the same as before and the results are shown in Fig. 7. In this case intermittent sliding began at $\tan\alpha = 0.11$. Transforming Equation [41] gives

$$\frac{\delta}{2L} = -\xi_y = \frac{\mu(4-\nu)}{2(1-\nu)}\,\frac{a}{D}\left[1-\left(1-\frac{\tan\alpha}{\mu}\right)^{1/2}\right] \quad ..[44]$$

which is plotted for comparison, putting $\mu = 0.11$.

The experiment was repeated with the surface flooded with a normal lubricating oil. It will be seen from Fig. 7 that the coefficient of limiting friction has been reduced by the lubricant, but only from 0.11 to 0.09, although it is evident, from the low value obtained dry, that grease films were still present. Thus the addition of a lubricating oil at rolling velocities too small to introduce any hydrodynamic action, has little effect upon the creep process.

Comparing the experimental results with the theoretical relationships of Equations [43] and [44], reasonably good agreement is found, particularly for small values of $T(T \ll \mu N)$ and again just before sliding begins $(T \rightarrow \mu N)$. The discrepancy is most marked in the intermediate range and is greatest at $T \doteqdot 0.75\mu N$ when the theoretical creep is about 25 per cent less than the measured value. The principal assumption underlying the theoretical treatment is that the locked region is circular and tangential to the circle of contact at the leading point $(-a, 0)$. It has been pointed out already that this assumption cannot be exact since it leads to slip velocities which are inconsistent with the direction of the assumed traction over the shaded area shown in Fig. 3(b). This area of error, although never large, reaches its maximum value when $a' \doteqdot 0.6\ a$; i.e., when $T \doteqdot 0.72\mu N$. It would seem reasonable to conclude that the departure of the true locked region from its assumed circular shape is the principal reason for the discrepancy between the experimental and theoretical results.

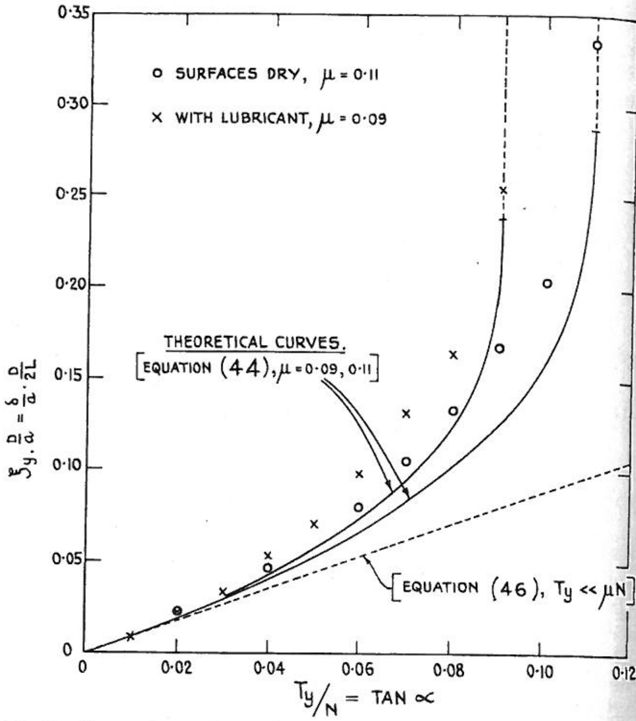

Fig. 7 Comparison of experimental transverse creep measurements with theoretical analysis

APPENDIX

Limiting Solution for Small Tangential Forces ($T \ll \mu N$)

If the tangential force is small compared with the limiting friction force, the expressions for the creep, Equations [37] and [41], reduce to

$$\xi_x = -\frac{(4 - 3\nu)}{16Ga^2} T_x \qquad [45]$$

$$\xi_y = -\frac{(4 - \nu)}{16Ga^2} T_y \qquad [46]$$

Under these circumstances the slip is confined to a thin "new moon" at the trailing edge of the circle of contact.

It is of interest to examine the limit to which the corresponding tractions tend when $T \ll \mu N$. With the aid of Equation [35] the tractions $X + X'$ of Equations [22] and [30] may be written in the form

$$X + X' = \frac{3T_x}{2\pi} \frac{(a^2 - r^2)^{1/2} - (a'^2 - r'^2)^{1/2}}{a^3 - a'^3} \qquad [47]$$

Now the limit of this distribution of traction as a' is made to approach a (i.e., $T_x \rightarrow 0$) may be shown without difficulty to be the traction

$$X = \frac{T_x}{2\pi a^2} \frac{a + x}{(a^2 - r^2)^{1/2}} \qquad [48]$$

Similarly, in the case of a transverse force the traction becomes

$$Y = \frac{T_y}{2\pi a^2} \frac{a + x}{(a^2 - r^2)^{1/2}} \qquad [49]$$

Rather surprisingly, the magnitude of these limiting tractions rises to infinity on the boundary of the circle of contact except for the point $(-a, 0)$ where it is zero. Physically, this zero value corresponds to the imposed condition that there should be no slip at this point.

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