3.4 Laminate Theory

MAC/GMC includes the capability to analyze general (symmetric and nonsymmetric) composite laminates [9], see Fig. 4. Mid-plane strains and resultant forces in the plane of the laminate may be applied. That is, the global laminate stress-strain relation that is solved within **MAC/GMC** is expressed as,

$$\begin{bmatrix} \bar{N}_{XX} \\ \bar{N}_{YY} \\ \bar{N}_{XY} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\gamma}_{xy} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \bar{\kappa}_{xx} \\ \bar{\kappa}_{yy} \\ \bar{\kappa}_{xy} \end{bmatrix} - \begin{bmatrix} \bar{N}^I_{XX} \\ \bar{N}^I_{YY} \\ \bar{N}^I_{XY} \end{bmatrix} - \begin{bmatrix} \bar{N}^T_{XX} \\ \bar{N}^T_{YY} \\ \bar{N}^T_{XY} \end{bmatrix} \tag{EQ 15}$$

$$\begin{bmatrix} \bar{M}_{XX} \\ \bar{M}_{YY} \\ \bar{M}_{XY} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\gamma}_{xy} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \bar{\kappa}_{xx} \\ \bar{\kappa}_{yy} \\ \bar{\kappa}_{xy} \end{bmatrix} - \begin{bmatrix} \bar{M}^I_{XX} \\ \bar{M}^I_{YY} \\ \bar{M}^I_{XY} \end{bmatrix} - \begin{bmatrix} \bar{M}^T_{XX} \\ \bar{M}^T_{YY} \\ \bar{M}^T_{XY} \end{bmatrix} \tag{EQ 16}$$

or

$$\begin{bmatrix} \bar{\underset{\sim}{N}} \\ \bar{\underset{\sim}{M}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{D} \end{bmatrix} \begin{bmatrix} \bar{\underset{\sim}{\varepsilon}} \\ \bar{\underset{\sim}{\kappa}} \end{bmatrix} - \begin{bmatrix} \bar{\underset{\sim}{N}}^I \\ \bar{\underset{\sim}{M}}^I \end{bmatrix} - \begin{bmatrix} \bar{\underset{\sim}{N}}^T \\ \bar{\underset{\sim}{M}}^T \end{bmatrix} \tag{EQ 17}$$

where $\bar{\underset{\sim}{N}}$, $\bar{\underset{\sim}{N}}^I$, $\bar{\underset{\sim}{N}}^T$, and $\bar{\underset{\sim}{M}}$, $\bar{\underset{\sim}{M}}^I$, $\bar{\underset{\sim}{M}}^T$ are the global laminate total, inelastic and thermal force and moment resultants, respectively. The matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{D}$ are the global laminate extensional, coupling and bending stiffnesses, respectively, and, $\bar{\underset{\sim}{\varepsilon}}$ and $\bar{\underset{\sim}{\kappa}}$ the global laminate mid-plane strain and mid-plane curvature, respectively.

In forming the laminate extensional stiffness $\underset{\sim}{A}$ the generalized method of cells model, **GMC**, is utilized to calculate the individual lamina properties. In this regard, the individual laminate stiffness, in lamina coordinates, $\underset{\sim}{Q}$ is given by,

$$\underset{\sim}{Q} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \tag{EQ 18}$$

in which the components of $\underset{\sim}{Q}$ are given as,

$$Q_{11} = C_{11} - \frac{C_{13}C_{31}}{C_{33}} \qquad Q_{12} = C_{12} - \frac{C_{13}C_{23}}{C_{33}}$$
$$Q_{22} = C_{22} - \frac{C_{23}C_{32}}{C_{33}} \qquad Q_{33} = C_{66} \tag{EQ 19}$$

The C_{ij} in the above are the effective macro properties for the unidirectional composite lamina obtained from **GMC**.

It follows, employing a Kirchhoff-Love hypothesis, that the lamina stress-strain relation in global (laminate) coordinates denoted by x-y is given by the relation,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_k \left[\begin{bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\gamma}_{xy} \end{bmatrix} + z \begin{bmatrix} \bar{\kappa}_{xx} \\ \bar{\kappa}_{yy} \\ \bar{\kappa}_{xy} \end{bmatrix} - \begin{bmatrix} \varepsilon^I_{xx} \\ \varepsilon^I_{yy} \\ \gamma^I_{xy} \end{bmatrix}_k - \begin{bmatrix} \varepsilon^T_{xx} \\ \varepsilon^T_{yy} \\ \gamma^T_{xy} \end{bmatrix}_k \right] \tag{EQ 20}$$

or

$$\underset{\sim}{\sigma}_k = \overline{\underset{\sim}{Q}_k}(\underset{\sim}{\bar{\varepsilon}} + z\underset{\sim}{\bar{\kappa}} - \underset{\sim}{\varepsilon}^I_k - \underset{\sim}{\varepsilon}^T_k) \tag{EQ 21}$$

where

$$\overline{\underset{\sim}{Q}_k} = R_k^{-1}\underset{\sim}{Q}_k R_k \tag{EQ 22}$$

$$R_k = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & (-2)\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{EQ 23}$$

and θ is the orientation of the longitudinal lamina axis with respect to the global x-direction, see Fig. 4, thus $\bar{\underset{\sim}{Q}}_k$ is the transformed lamina stiffness, i.e. from local lamina to global laminate coördinates. In addition, $\underset{\sim}{\sigma}_k$ is the lamina stress vector in laminate coordinates. It then follows that the global laminate extensional stiffness $\underset{\sim}{A}$ is given by,

$$\underset{\sim}{A} = \sum_{k=1}^{nly} \bar{\underset{\sim}{Q}}_k t_k \tag{EQ 24}$$

in which nly is the total number of layers in the laminate and t_k is the thickness of the k^{th} lamina. The coupling and bending stiffnesses can be similarly developed and given by the following expressions:

$$\underset{\sim}{B} = \frac{1}{2}\sum_{k=1}^{nly} \underset{\sim}{\bar{Q}}_k (z_k^2 - z_{k-1}^2) \qquad \text{(EQ 25)}$$

$$\underset{\sim}{D} = \frac{1}{3}\sum_{k=1}^{nly} \underset{\sim}{\bar{Q}}_k (z_k^3 - z_{k-1}^3) \qquad \text{(EQ 26)}$$

where z_k is the distance (considering the sign) to the top of layer k from the mid-plane.

Returning to EQ. 17, the quantities $\underset{\sim}{\bar{N}}^I$ and $\underset{\sim}{\bar{N}}^T$ (the laminate inelastic and thermal force resultants, respectively) are calculated from the individual lamina contributions through the following relations,

$$\underset{\sim}{\bar{N}}^I = \sum_{k=1}^{nly} \underset{\sim}{\bar{Q}}_k \int_{z_{k-1}}^{z_k} \underset{\sim}{\varepsilon}_k^I dz \qquad \underset{\sim}{\bar{M}}^I = \sum_{k=1}^{nly} \underset{\sim}{\bar{Q}}_k \int_{z_{k-1}}^{z_k} \underset{\sim}{\varepsilon}_k^I z dz \qquad \text{(EQ 27)}$$

and

$$\underset{\sim}{\bar{N}}^T = \sum_{k=1}^{nly} \underset{\sim}{\bar{Q}}_k \underset{\sim}{\varepsilon}_k^T t_k \qquad \underset{\sim}{\bar{M}}^T = \frac{1}{2}\sum_{k=1}^{nly} \underset{\sim}{\bar{Q}}_k \underset{\sim}{\varepsilon}_k^T (z_k^2 - z_{k-1}^2) \qquad \text{(EQ 28)}$$

where the integrals in EQ. 27 are performed using second order gauss quadrature which requires two integration points per layer. Thus all field quantities are tracked at the two gauss quadrature points in each layer of the laminate in **MAC/GMC.**

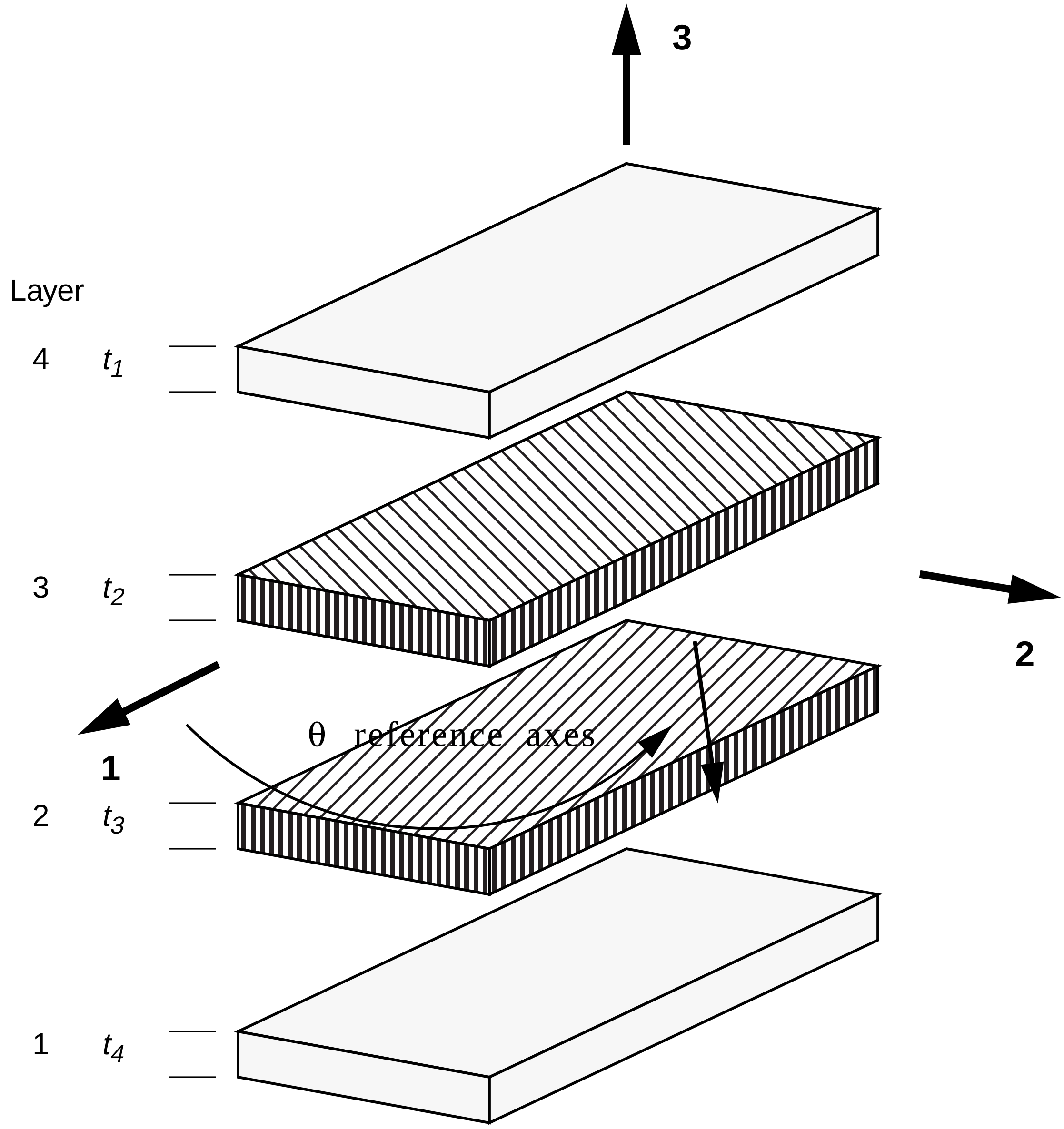

Figure 4: Laminate Coordinate System