3.3 Available Constituent Constitutive Models

Currently **MAC/GMC** provides two elastic and five inelastic constitutive models. These models have been selected purely based upon the availability of material parameters for the materials of interest. However, **MAC/GMC** is designed in a modular fashion thus allowing the implementation of additional constitutive models through a user defined subroutine. Two of the five available inelastic models are capable of representing transversely isotropic material behavior, thus allowing one to investigate the reinforcement of an anisotropic matrix allowing idealization of a heterogeneous material via a pseudo-homogenous anisotropic material (e.g. fiber tow). In all five inelastic models a purely elastic response is possible by modifying a single material parameter for each model as noted below.

3.3.1 Transversely Isotropic Elastic Model

Reference: Jacob Aboudi, Mechanics of Composite Materials, Elsevier, 1991

The following transversely isotropic model is provided for those materials that have an elastic only response and whose strong direction is aligned with the 1 axis shown in Fig. 6, such as the fiber constituent in a composite.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\left(\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix} - \begin{bmatrix}\alpha_L\Delta T\\ \alpha_T\Delta T\\ \alpha_T\Delta T\\ 0\\ 0\\ 0\end{bmatrix}\right)$$

where components C_{ij} can be expressed in terms of five independent constants,

$$E_A, E_T, \nu_A, \nu_T, G_A$$

thus,

$$C_{11} = E_A + 4\kappa \nu_A^{\ 2}$$
$$C_{12} = 2\kappa \nu_A$$
$$C_{22} = \kappa + \frac{0.5 E_T}{(1+\nu_T)}$$
$$C_{23} = \kappa - \frac{0.5 E_A}{(1+\nu_T)}$$
$$C_{44} = G_A$$
$$C_{66} = \frac{(C_{22} - C_{23})}{2}$$

with

$$\kappa = 0.25 E_A / [0.5(1-\nu_T)(E_A/E_T) - \nu_A^{\ 2}]$$

3.3.2 Anisotropic Elastic Model

Reference: S. M. Arnold, A Transversely Isotropic Thermoelastic Theory, NASA TM 101302, 1988

An alternative transversely isotropic model has also been provided for those materials that have an elastic only response, but whose strong direction may not be aligned with the 1 axis shown in Fig. 6 (i.e., the plane of isotropy is allowed to rotate), for example in the case of woven composites, see **Example N**. This elastic model is consistent with the anisotropic inelastic models described subsequently and is defined within the global coordinate system via a direction cosine vector, d_i.

$$\underset{\sim}{\sigma} = [\underset{\sim}{C}]\{\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^{th}\}$$

where the stiffness matrix, $\underset{\sim}{C}$, is a function of the five independent material parameters $E_A, E_T, \nu_A, \nu_T, G_A$ and the direction cosine vector, d_i, and, in general is fully populated. Note the multiaxial thermal strain tensor is assumed to have the following form,

$$\varepsilon_{ij}^{th} = [(\alpha_L - \alpha_T) d_i d_j + \delta_{ij} \alpha_T] \Delta T$$

Further details can be found in the above reference.

3.3.3 Bodner-Partom Model

Reference: Jacob Aboudi, Mechanics of Composite Materials, Elsevier, 1991

This model represents a Bodner-Partom viscoplastic material with isotropic hardening, Z, and can be used for an initially isotropic metallic material.

The flow law is given as:

$$\dot{\varepsilon}_{ij}^{I} = \Lambda s_{ij}$$

where

$$\Lambda = \sqrt{\frac{D_2^{PL}}{J_2}}$$

$$D_2^{PL} = D_0^2 exp\left[-\left(\frac{A^2}{J_2}\right)^n\right]$$

$$A^2 = \frac{1}{3}Z_{eff}^2\left(\frac{n+1}{n}\right)^{\frac{1}{n}}$$

$$J_2 = \frac{1}{2}S_{ij}S_{ij}$$

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

The evolution law for isotropic hardening is given as:

$$\dot{Z} = m(Z_1 - Z_{eff})\frac{\dot{W}^{PL}}{Z_0}$$

where Z_0, Z_1 and m are inelastic constants and the plastic work rate, $\dot{W}^{PL}$, is given by;

$$\dot{W}^{PL} = \sigma_{ij}\dot{\varepsilon}_{ij}^{I}$$

$$Z_{eff} = Z_0 + q\int_0^t \dot{Z}(\tau)d\tau + (1-q)\sum_{i,j=1}^{3} r_{ij}\int_0^t \dot{Z}(\tau)r_{ij}(\tau)d\tau$$

$$r_{ij}(t) = \sigma_{ij}(t)/[\sigma_{kl}(t)\sigma_{kl}(t)]^{1/2}$$

An elastic only response may be obtained by setting the material parameter D_0 to zero.

3.3.4 Modified Bodner-Partom Model

Reference: R. W. Neu, "Nonisothermal Material Parameters For the Bodner-Partom Model", MD-Vol. 43, Material Parameter Estimation for Modern Constitutive Equations, L. A. Betram, S.B. Brown, and A.D. Freed, Eds., ASME, Book No. H00848, 1993.

This model represents a nonisothermal Bodner-Partom viscoplastic material with isotropic, Z^I, and directional hardening, Z^D, and can be used for an initially isotropic metallic material.

The flow law is given as:

$$\dot{\varepsilon}_{ij}^{I} = \Lambda s_{ij}$$

where

$$\Lambda = D_0 exp\left[-0.5\left(\frac{Z^2}{3J_2}\right)^n\right]\frac{1}{\sqrt{J_2}}$$

$$Z = Z^I + Z^D$$

$$J_2 = \frac{1}{2}S_{ij}S_{ij}$$

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

The evolution law for the isotropic hardening, Z^I, is given as:

$$\dot{Z}^I = m_1(Z_1 - Z^I)\dot{W}^{PL} - A_1 Z_1\left(\frac{Z^I - Z_2}{Z_1}\right)^{r_2} + \left(\left(\frac{Z^I - Z_2}{Z_1 - Z_2}\right)\frac{\partial Z_1}{\partial T} + \left(\frac{Z_2 - Z^I}{Z_1 - Z_2}\right)\frac{\partial Z_2}{\partial T}\right)\dot{T}$$

where Z_0, Z_1, Z_2, A_1, r_1 and m_1 are the material parameters associated with isotropic hardening. The magnitude of the directional hardening is defined as the scalar product of a state variable, β_{ij}, and unit stress vector, u_{ij}, as given below.

$$Z^D = \beta_{ij}u_{ij}$$

$$\dot{\beta}_{ij} = m_2(Z_3 u_{ij} - \beta_{ij})\dot{W}^{PL} - A_1 Z_1\left(\frac{\sqrt{\beta_{ij}\beta_{ij}}}{Z_1}\right)^{r_2} v_{ij} + \left(\left(\frac{\beta_{ij}}{Z_3}\right)\frac{\partial Z_3}{\partial T}\right)\dot{T}$$

$$u_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{kl}\sigma_{kl}}}$$

$$v_{ij} = \frac{\beta_{ij}}{\sqrt{\beta_{kl}\beta_{kl}}}$$

where, the plastic work rate is defined as; $\dot{W}^{PL} = \sigma_{ij}\dot{\varepsilon}_{ij}^{I}$ and with, Z_3, A_2, r_2 and m_2 being the material parameters defining the directional evolution. An elastic only response may be obtained by setting the material parameter D_0 to zero.

3.3.5 Robinson Viscoplastic Model

Reference: S.M. Arnold, D.N. Robinson, and P.A. Bartlotta, "Unified Viscoplastic Behavior of Metal Matrix Composites", NASA TM 105819, 1992

This model represents a **transversely isotropic material** wherein the vector of direction cosines, d_i, defines the preferred material direction. In this model the strength of anisotropy is specified by the parameters ω and η; where ω is the ratio of the normal longitudinal and transverse yield stress and η is the ratio of longitudinal and transverse threshold shear stress.

Flow Law:

$$\dot{\varepsilon}_{ij}^{I} = \frac{\langle F^n \rangle}{2\mu}\Gamma_{ij}$$

Evolution Law:

$$\dot{a}_{ij} = \frac{H}{G^{\beta}}\dot{\varepsilon}_{ij}^{I} - RG^{m-\beta}\Pi_{ij}$$

where

$$\Gamma_{ij} = \Sigma_{ij} - \xi(D_{ki}\Sigma_{jk} + D_{jk}\Sigma_{ki} - 2I_0 D_{ij}) - \frac{1}{2}\zeta I_0(3D_{ij} - \delta_{ij})$$

$$\Pi_{ij} = a_{ij} - \xi(D_{ki}a_{jk} + D_{jk}a_{ki} - 2\hat{I}_0 D_{ij}) - \frac{1}{2}\zeta\hat{I}_0(3D_{ij} - \delta_{ij})$$

and

$$F = \frac{1}{\kappa_T^2}\left[I_1 + \frac{1}{\eta^2}I_2 + \frac{9}{4(4\omega^2-1)}I_3\right] - 1$$

$$\hat{G} = \frac{1}{\kappa_T^2}\left[\hat{I}_1 + \frac{1}{\eta^2}\hat{I}_2 + \frac{9}{4(4\omega^2-1)}\hat{I}_3\right]$$

$$G = \langle \hat{G} - \hat{G}_0 \rangle Hv[S_{ij}\pi_{ij}] + \hat{G}_0$$

$$I_1 = J_2 - I - \frac{1}{4}I_3 \qquad I_2 = I - I_3 \qquad I_3 = I_0^2$$

$$J_2 = \frac{1}{2}\Sigma_{ij}\Sigma_{ji} \qquad I = D_{ij}\Sigma_{ji} \qquad D_{ij} = d_i d_j \qquad \Sigma_{ij} = S_{ij} - a_{ij}$$

$$\xi = \frac{\eta^2 - 1}{\eta^2} \qquad \zeta = \frac{4(\omega^2-1)}{4\omega^2 - 1}$$

with $\kappa_T, \mu, n, H, \beta, R, m, G_o, \eta$ and ω representing the associated required inelastic material parameters and < > denoting a Macauley bracket, as defined below.

$$\langle\ \rangle = \left\{ \begin{array}{ll} 0 & x < 0 \\ x & x \geq 0 \end{array} \right\}$$

The invariants $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_3$ are the same I_1, I_2, I_3 as those given above but with Σ_{ij} replaced by a_{ij}. Special cases involving an isotropic material and/or elastic only response can be obtained by defining $\omega = \eta = 1$ and/or by setting κ_T to an extremely large number.

3.3.6 Generalized Viscoplastic Potential Structure (GVIPS) Model

References: S.M. Arnold, A.F. Saleeb, and M.G. Castelli, “A Fully Associative, Non-Linear Kinematic, Unified Viscoplastic Model for Titanium Based Matrices”, NASA TM 106609, 1994.

S.M. Arnold, A.F. Saleeb, and M.G. Castelli, “A Fully Associative, Nonisothermal, NonLinear Kinematic, Unified Viscoplastic Model for Titanium Alloys”, NASA TM 106926, 1994.

This model is a fully associative, multiaxial, nonlinear kinematic hardening viscoplastic model for use with initially isotropic metallic materials. A unique aspect of this model is the inclusion of non-linear hardening through the use of a compliance operator Q_{ijkl} in the evolution law for the back stress. This non-linear tensorial operator is significant in that it allows both the flow and evolutionary laws to be fully associative (and therefore easily integrated) and greatly influences the multiaxial response under non-proportional loading paths.

Flow Law:

$$\dot{\varepsilon}^I_{ij} = \frac{3}{2}\left\|\dot{\varepsilon}^I_{ij}\right\|\frac{\Sigma_{ij}}{\sqrt{J_2}} \qquad if \qquad F \geq 0$$

where

$$\left\|\dot{\varepsilon}^I_{ij}\right\| = \sqrt{\frac{2}{3}\dot{\varepsilon}^I_{ij}\dot{\varepsilon}^I_{ij}} = \frac{\mu F^n}{\kappa}$$

Internal constitutive rate equation:

$$\dot{a}_{ij} = L_{ijrs}\dot{A}_{rs}$$

Evolution Law:

$$\dot{A}_{rs} = \dot{\varepsilon}^I_{rs} - \frac{3}{2}\frac{\beta\kappa}{\kappa_o^2}\left\|\dot{\varepsilon}^I_{ij}\right\|\frac{a_{rs}}{\sqrt{G}}Hv[Y] - \frac{3R_\alpha B_0 G^q}{\kappa_o{}^2}a_{rs} \qquad if \qquad a_{ij}\Sigma_{ij} \geq 0$$

$$\dot{A}_{rs} = Q_{rslm}E_{lmnp}\left(\dot{\varepsilon}^I_{np} - \frac{3}{2}\frac{\beta\kappa}{\kappa_o^2}\left\|\dot{\varepsilon}^I_{ij}\right\|\frac{a_{np}}{\sqrt{G}}Hv[Y] - \frac{3R_\alpha B_0 G^q}{\kappa_o{}^2}a_{np}\right) \qquad if \qquad a_{ij}\Sigma_{ij} < 0$$

where

$$F = \langle \frac{\sqrt{J_2}}{\kappa} - Y \rangle$$

$$Y = \langle 1 - \beta\sqrt{G} \rangle$$

$$G = \frac{I_2}{\kappa_o^{\;2}}$$

$$L_{ijrs} = [Q_{ijrs}]^{-1} = \frac{\kappa_o^2}{3B_0(1 + B_1 p G^{p-1})}\left(I_{ijrs} - \frac{3B_1 p(p-1)G^{p-2}}{\kappa_o^2(1 + B_1 p G^{p-1}(6p-5))} a_{rs}a_{ij} \right)$$

and

$$I_2 = \frac{3}{2}a_{ij}a_{ij} \qquad \Sigma_{ij} = S_{ij} - a_{ij}$$

$$J_2 = \frac{3}{2}\Sigma_{ij}\Sigma_{ij} \qquad S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

with $\kappa, \mu, n, \kappa_o, B_0, B_1, p, R_\alpha, q$ and β being the associated required inelastic material parameters. Typically, $\kappa, \mu, B_0, R_\alpha$ and β are taken to be functions of temperature and $\kappa_o = \kappa(T_{ref})$ is the initial drag stress at the reference temperature. The special case of an elastic only response maybe obtained by setting κ to an extremely large value.

3.3.7 Transversely Isotropic GVIPS Model (TGVIPS)

Reference: A.F. Saleeb and T.E. Wilt, Analysis of the Anisotropic Viscoplastic-Damage Response of Composite Laminates-Continuum Basis and Computational Algorithms, Int. J. Numer. Meth. Enging., Vol. 36, pp. 1629-1660, 1993.

This model is a fully associative, multiaxial, isothermal, nonlinear kinematic hardening viscoplastic model for use with **initially transversely isotropic** metallic materials. A unique aspect of this model is the inclusion of non-linear hardening through the use of a compliance operator Q_{ijkl} in the evolution law for the back stress. This non-linear tensorial operator is significant in that it allows both the flow and evolutionary laws to be fully associative (and therefore easily integrated) and greatly influences the multiaxial response under non-proportional loading paths.

The flow law for the inelastic strain, $\dot{\underset{\sim}{\varepsilon}}^{I}$, is given by,

$$\dot{\underset{\sim}{\varepsilon}}^{I} = \frac{\langle F^{n} \rangle}{2\mu} \underset{\sim}{\Gamma}$$

and the evolution law for internal stress, $\underset{\sim}{\alpha}$ is given by,

$$\dot{\underset{\sim}{\alpha}} = \left[\underset{\sim}{Z}_{m} + \frac{h'}{h(1+2\beta)} (\underset{\sim}{\alpha} \otimes \underset{\sim}{\alpha}) \right] \left(\frac{H}{G^{\beta}} \dot{\underset{\sim}{\varepsilon}}^{I} - R G^{m} \underset{\sim}{\Pi} \right)$$

where $h = H/G^{\beta}$, $h' = -\beta/G$ and $Z_{m} = \underset{\sim}{M}^{-1}$

$$F = \frac{1}{2\kappa_{t}^{2}} (\underset{\sim}{\sigma} - \underset{\sim}{\alpha}) : \underset{\sim}{M} : (\underset{\sim}{\sigma} - \underset{\sim}{\alpha}) - 1$$

$$G = \frac{1}{2\kappa_{t}^{2}} \underset{\sim}{\alpha} : \underset{\sim}{M} : \underset{\sim}{\alpha}$$

$$\underset{\sim}{\Gamma} = \underset{\sim}{M} : (\underset{\sim}{\sigma} - \underset{\sim}{\alpha})$$

$$\underset{\sim}{\Pi} = \underset{\sim}{M} : \underset{\sim}{\alpha}$$

$$\underset{\sim}{M} = \underset{\sim}{P} - \xi \underset{\sim}{Q} - \frac{1}{2} \zeta \underset{\sim}{R}$$

The anisotropy of the material is introduced through the $\underset{\sim}{M}$ matrix, specifically the parameters, ζ, ξ which are defined as,

$$\xi = \frac{\eta^2 - 1}{\eta^2} \qquad \zeta = \frac{4(\omega^2 - 1)}{4\omega^2 - 1}$$

$$\eta = \frac{K_l}{K_t} \qquad \omega = \frac{Y_l}{Y_t}$$

In the above, $0 \le \xi \le 1$ and $0 \le \zeta \le 1$ are the material strength ratios, in which the constants K_l (K_t) are the threshold strengths in longitudinal (transverse) shear, and Y_l (Y_t) are the threshold strengths in longitudinal (transverse) tension. Note that for an isotropic material, $\omega = \eta = 1$.

In addition, the fourth-order tensors $\underset{\sim}{P}$, $\underset{\sim}{Q}$, and $\underset{\sim}{R}$ are defined as,

$$\underset{\sim}{P} = \underset{\sim}{I} - \frac{1}{3}(\underset{\sim}{\delta} \otimes \underset{\sim}{\delta}) \qquad I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{ji} + \delta_{il}\delta_{jk})$$

$$Q_{ijkl} = \frac{1}{2}(D_{ik}\delta_{jl} + D_{il}\delta_{jk} + D_{jk}\delta_{il} + D_{jl}\delta_{ik}) - 2D_{ij}D_{kl}$$

$$R_{ijkl} = 3D_{ij}D_{kl} - (D_{ij}\delta_{kl} + \delta_{ij}D_{kl}) + \frac{1}{3}(\delta_{ij}\delta_{kl})$$

In the above, the vector of direction cosines d_i defines the orientation of the material fiber, which leads to the material directionality tensor $\underset{\sim}{D}$, ($\underset{\sim}{D} = d_i d_j$). Also, $\underset{\sim}{\delta}$ is the Kronecker delta (second-order identity tensor) and $\underset{\sim}{I}$ is the fourth-order identity tensor. Finally, the symbol **:** indicates double-contraction and $\otimes$ cross product.

☞ **Note:** When calculating Z_m for the three dimension case one needs to replace $\underset{\sim}{P}$ with $\hat{\underset{\sim}{P}}$ = diag[1,1,1,2,2,2] as $\underset{\sim}{P}$ is singular for the three dimensional case.